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LUENBERGER AND MALMQUIST PRODUCTIVITY INDICES: THEORETICAL COMPARISONS AND EMPIRICAL ILLUSTRATION

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ABSTRACT

This contribution establishes, from a theoretical viewpoint, the relations between the Malmquist productivity indices, that measure in either input or output orientations, and the Luenberger productivity indices, that can simultaneously contract inputs and expand outputs, but that can also measure in either input or output orientations. The main result is that a Malmquist productivity index overestimates productivity changes, since it provides productivity measures that are nearly twice those given by the Luenberger productivity index looking for simultaneous contractions of inputs and expansions of outputs. This relationship is empirically illustrated using data from 20 OECD countries over the 1974–97 period.

Keywords: technical change, productivity growth, Malmquist index, Luenberger index, proportional distance, function
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I. INTRODUCTION

For many years, productivity growth measures have identified total factor productivity growth with a shift in technology. Productivity growth measures have been evaluated via continuous time production functions on macro- or micro-economic data, whereby output variations that are left unexplained by input variations — the famous Solow residual — are interpreted as technological change. In the last two decades, there is growing awareness that ignoring inefficiency in input usage or output production yields a biased measure of productivity growth.

Recently, a technology-based, discrete-time Malmquist productivity index, initially defined by Caves, Christensen and Diewert (1982a, b) as a ratio of distance functions, has become increasingly popular, since it offers a more complete picture of productivity growth.[1] This is mainly due to the innovations of Färe *et al.* (1995) who showed how: (i) to relax the implicit hypothesis of technical efficiency maintained in Caves, Christensen and Diewert (1982a, b); (ii) to decompose this index into technical efficiency changes and technology shifts; and (iii) to compute the Malmquist index relative to multiple inputs and outputs non-parametric technologies by exploiting the inverse relationship between output distance functions and output-oriented technical efficiency measures.

The advantage of using non-parametric frontier technologies is that they impose no *a priori* functional form on technology, nor any restrictive assumptions regarding input remuneration. Furthermore, the frontier nature of these technologies allows capturing any productive inefficiency and offers a "benchmarking" perspective. In a macro-economic context, for instance, an economy's performance can be evaluated with respect to both its past experience and the best practices established by other countries. For a unified discussion of efficiency and productivity from an index theory perspective and its methodological advantages, the reader can consult Chambers, Färe and Grosskopf (1994).

One remaining limitation for this primal Malmquist productivity index was that one had to choose between either an output- or an input-oriented perspective corresponding to whether one assumes revenue maximization or cost minimization as the proper behavioural goal for the sample at hand (Färe, Grosskopf and Lovell, 1985).[2] But, since

[1] This productivity index is named after Malmquist (1953), who actually defined a similar quantity index of distance functions in a consumer context. Caves, Christensen and Diewert (1982a, b) approximate the Malmquist productivity index by a less general Törnqvist index, lacking estimation procedures for distance functions at the time.

[2] Note that on constant returns to scale technologies, which are predominantly used for computing Malmquist productivity indices, this choice of orientation makes no difference. It suffices to add that this choice of technology itself is the source of some controversy: see, e.g., Färe, Grosskopf and Roos (1998).

no distance function was available that was dual to the profit function, no corresponding Malmquist productivity index could be defined.

By introducing the shortage function, Luenberger (1992a, b) generalizes existing distance functions and provides a flexible tool to account for both input contractions and output improvements when measuring efficiency.[3] As shown by Luenberger (1992b, 1995) and Chambers, Chung and Färe (1998), this shortage function (directional distance function in the terminology of Chambers, Chung and Färe, 1998) is dual to the profit function. Making use of the shortage function, Chambers, Färe and Grosskopf (1996) introduced the Luenberger productivity index, as a difference of directional distance functions (see also Balk, 1998; Chambers, 2002).

This primal Luenberger productivity index can probably best be interpreted in the context of recent attempts to develop test and economic approaches to index number theory based on differences rather than more traditional ratios. While economics as a discipline has long been used to work with ratios, the business and accounting community is clearly more familiar with analysing, e.g., cost, revenue or profit differences. Apart from tradition, the ratio and difference approaches to index numbers also differ in terms of certain basic properties of great practical significance. While ratios are unit invariant, differences are not. But differences are invariant to changes in the origin, while ratios are not. Furthermore, ratios have difficulties coping with zero observations, while this poses little problem for differences. For a systematic discussion of both ratio and difference approaches to index number theory from both a test and an economic perspective, the reader is referred to Chambers (1998, 2002) and Diewert (1998, 2000), among others.[4]

The Luenberger productivity index, as a generalization of the Malmquist index, is required for evaluating organizations that can be assumed to be profit maximizing. Furthermore, this Luenberger productivity index can specialize to an output- or input-oriented perspective corresponding to the revenue maximization and cost minimization cases when necessary. Clearly, the Luenberger productivity indices encompass the Malmquist productivity approach.

This paper outlines the theoretical link between the Malmquist and Luenberger productivity indices and provides an empirical illustration of the relation between these two measures for 20 OECD countries over the 1974–97 period. In particular, we show that the Malmquist productivity index overestimates the true productivity change measured via the

[3] Furthermore, the directional distance function complies with the mathematical notion of a distance.

[4] Some of these authors suggest the term "indicators" for measures based on differences and reserve the term "indices" for measures defined as ratios. Lacking convergence on this semantic issue, we stick to tradition.

Luenberger productivity index. The contribution is structured as follows. Section II lays out the basic framework by providing definitions of the various distance functions and productivity indices. The next section states two propositions with the main results of this contribution. An empirical illustration of the main relation between the Malmquist and Luenberger productivity indices is reported in Section IV. Section V concludes.

II. TECHNOLOGY, DISTANCE FUNCTIONS AND PRODUCTIVITY INDICES

This section introduces the assumptions on the production possibility set and the methods used to estimate the components of total factor productivity. In the case of the Malmquist productivity index one employs Shephard distance functions, while the Luenberger index estimation makes use of a proportional distance function.

Production technology transforms inputs $x = (x_1, \ldots, x_n) \in R^n_+$ into outputs $y = (y_1, \ldots, y_p) \in R^p_+$. In each time period t, the set of all feasible input and output vectors is called the production possibility set, $T(t)$, and is defined as follows:

$$T(t) = \left\{(x^t, y^t) \in R^{n+p}_+; x^t \text{ can produce } y^t\right\}. \tag{1}$$

It satisfies the following assumptions: (T.1) $(0, 0) \in T(t)$, $(0, y^t) \in T(t) \Rightarrow y^t = 0$, i.e., no outputs without inputs; (T.2) the set $A(x^t) = \{(u^t, y^t) \in T(t); u^t \leq x^t\}$ of dominating observations is bounded $\forall x^t \in R^n_+$, i.e., infinite outputs are not allowed with a finite input vector; (T.3) $T(t)$ is a closed set; (T.4) $\forall (x^t, y^t) \in T(t)$, $(x^t, -y^t) \leq (u^t, -v^t) \Rightarrow (u^t, v^t) \in T(t)$, i.e., fewer outputs can always be produced with more inputs, and inversely; (T.5) $T(t)$ is convex.

The estimation of efficiency relative to production frontiers relies on the theory of distance or gauge functions. In economics, distance functions are related to the notion of the "coefficient of resource utilization" due to Debreu (1951) and to the efficiency measures introduced by Farrell (1957). The Debreu–Farrell efficiency measure $E_{T(t)}(x^t, y^t)$ is the inverse of the Shephard (1953) distance function. In the input-oriented case, this measure $E^i_{T(t)}(x^t, y^t)$ is based upon the minimum contraction of an input vector by a scalar λ to catch up the production frontier:

$$E^i_{T(t)}(x^t, y^t) = \min_{\lambda}\{\lambda; (\lambda x^t, y^t) \in T(t), \lambda \geq 0\}. \tag{2}$$

In the case of an output measure, $E^o_{T(t)}(x^t, y^t)$ is based upon the maximum expansion of an output vector by a scalar θ to catch up the production frontier, i.e., $E^o_{T(t)}(x^t, y^t) = \max_\theta\{\theta; (x^t, \theta y^t) \in T(t), \theta \geq 1\}$. Under constant returns to scale, Färe, Grosskopf and Lovell (1985) have shown that: $E^o_{T(t)}(x^t, y^t) = [E^i_{T(t)}(x^t, y^t)]^{-1}$.

Denoting $E^i_{T(b)}(x^a, y^a) = \min_\lambda\{\lambda;\ (\lambda x^a, y^a) \in T(b)\}$ where $(a,\ b) \in \{t,\ t+1\} \times \{t,\ t+1\}$, the input-oriented Malmquist index $M^i(x^t,\ y^t,\ x^{t+1},\ y^{t+1})$ is linked to the input Debreu–Farrell measures as follows:

$$M^i(x^t, y^t, x^{t+1}, y^{t+1}) = \left[\frac{E^i_{T(t)}(x^t, y^t)}{E^i_{T(t)}(x^{t+1}, y^{t+1})} \frac{E^i_{T(t+1)}(x^t, y^t)}{E^i_{T(t+1)}(x^{t+1}, y^{t+1})}\right]^{1/2}. \quad (3)$$

This is actually a geometric mean of period t (the first ratio) and period $t+1$ (the second ratio) Malmquist indices, in an effort to avoid an arbitrary selection among base years. Values below (above) unity reveal productivity growth (decline). In a similar way, one can define a Malmquist output productivity index as follows:

$$M^o(x^t, y^t, x^{t+1}, y^{t+1}) = \left[\frac{E^o_{T(t)}(x^t, y^t)}{E^o_{T(t)}(x^{t+1}, y^{t+1})} \frac{E^o_{T(t+1)}(x^t, y^t)}{E^o_{T(t+1)}(x^{t+1}, y^{t+1})}\right]^{1/2}. \quad (4)$$

Under constant returns to scale, clearly $M^o(x^t,\ y^t,\ x^{t+1},\ y^{t+1}) = [M^i\ (x^t,\ y^t,\ x^{t+1},\ y^{t+1})]^{-1}$.

Focusing again on the input-oriented Malmquist index, it can be geometrically decomposed into two components:

$$\begin{aligned} M^i(x^t, y^t, x^{t+1}, y^{t+1}) = {} & \frac{E^i_{T(t)}(x^t, y^t)}{E^i_{T(t+1)}(x^{t+1}, y^{t+1})} \\ & \cdot \left[\frac{E^i_{T(t+1)}(x^{t+1}, y^{t+1})}{E^i_{T(t)}(x^{t+1}, y^{t+1})} \frac{E^i_{T(t+1)}(x^t, y^t)}{E^i_{T(t)}(x^t, y^t)}\right]^{1/2}, \end{aligned} \quad (5)$$

whereby the first ratio (outside the square brackets) represents technical efficiency changes and the second geometric product of ratios (inside the square brackets) captures technological change.[5] A similar decomposition applies to the Malmquist output productivity index.

To define the Luenberger productivity index, we first discuss the proportional distance function D_T: $T \to R$ that involves simultaneous proportional input and output variations:

$$D_{T(t)}(x^t, y^t) = \max_\delta\{\ \delta \geq 0;\ ((1-\delta)x^t, (1+\delta)y^t) \in T(t)\}. \quad (6)$$

This is a special case of the shortage function (Luenberger, 1994) or directional distance function (Chambers, Färe and Grosskopf, 1996; Chambers, Chung and Färe, 1996). It is also a special case of the Farrell

[5] Further decompositions of both technical efficiency and technological change components have been proposed in the literature (see Färe, Grosskopf and Roos, 1998, for a survey).

proportional distance (Briec, 1997), a generalization of the Debreu–Farrell measure. Note that the directional distance function is defined using a general directional vector ($-g_i$, g_o), whereas we consider the special case $g_i = x$ and $g_o = y$. The axiomatic properties of this particular function are studied in Briec (1997) and Chambers, Chung and Färe (1998). An input-oriented version of the proportional distance function is defined as:

$$D^i_{T(t)}(x^t, y^t) = \max_{\delta}\{\, \delta \geq 0;\ ((1-\delta)x^t, y^t) \in T(t)\} = 1 - E^i_{T(t)}(x^t, y^t), \quad (7)$$

whereby the last expression reveals its relation with the Debreu–Farrell input efficiency measure (and, implicitly, with the input distance function). Similarly, an output-oriented version of the proportional distance function is defined as:

$$D^o_{T(t)}(x^t, y^t) = \max_{\delta}\{\, \delta \geq 0;\ (x^t, (1+\delta)y^t) \in T(t)\} = E^o_{T(t)}(x^t, y^t) - 1, \quad (8)$$

where the last part again establishes the relation with the output-oriented Debreu–Farrell efficiency measure.

The Luenberger productivity index $L(x^t, y^t, x^{t+1}, y^{t+1})$, initially proposed in Chambers, Färe and Grosskopf (1996) and Chambers (2002), is, in the case of proportional distance functions, defined as follows:

$$\begin{aligned} L(x^t, y^t, x^{t+1}, y^{t+1}) = \frac{1}{2}[&(D_{T(t)}(x^t, y^t) - D_{T(t)}(x^{t+1}, y^{t+1})) \\ &+ (D_{T(t+1)}(x^t, y^t) - D_{T(t+1)}(x^{t+1}, y^{t+1}))]. \end{aligned} \quad (9)$$

Again, to avoid an arbitrary choice between base years, an arithmetic mean of a difference-based Luenberger productivity index in base year t (first difference) and $t+1$ (second difference) has been taken (e.g., Balk, 1998, pp. 173–4)). Productivity growth (decline) is indicated by positive (negative) values. Notice that the distinction between difference and ratio approaches to productivity, and more in general to index number theory, relies on the structural difference between directional compared to traditional distance functions: the former have an additive structure, while the latter are multiplicative in nature.[6]

It is equally possible to define input- and output-oriented versions of this Luenberger productivity index based, respectively, on the input and the output proportional distance functions. The input variant of the Luenberger productivity index is defined as follows:

[6] Recently, Chambers (1998) defines difference-based Luenberger input and output indices founded upon a special case of the shortage (directional distance) function known as the translation function. These input indices are generalizations of the Malmquist input quantity index defined in Chambers, Färe and Grosskopf (1994).

$$L^i(x^t, y^t, x^{t+1}, y^{t+1}) = \frac{1}{2}[(D^i_{T(t)}(x^t, y^t) - D^i_{T(t)}(x^{t+1}, y^{t+1})) + (D^i_{T(t+1)}(x^t, y^t) - D^i_{T(t+1)}(x^{t+1}, y^{t+1}))], \quad (10)$$

while the output Luenberger productivity index can be defined as:

$$L^o(x^t, y^t, x^{t+1}, y^{t+1}) = \frac{1}{2}[(D^o_{T(t)}(x^t, y^t) - D^o_{T(t)}(x^{t+1}, y^{t+1})) + (D^o_{T(t+1)}(x^t, y^t) - D^o_{T(t+1)}(x^{t+1}, y^{t+1}))]. \quad (11)$$

The relationship between the three Luenberger indices is established in the next section. The input- and output-oriented versions of the Luenberger index are the difference-based alternatives to the similarly oriented Malmquist indices based on ratios.

The Luenberger productivity index is additively decomposed as follows:

$$L(x^t, y^t, x^{t+1}, y^{t+1}) = [D_{T(t)}(x^t, y^t) - D_{T(t+1)}(x^{t+1}, y^{t+1})] + \frac{1}{2}[(D_{T(t+1)}(x^{t+1}, y^{t+1}) - D_{T(t)}(x^{t+1}, y^{t+1})) + (D_{T(t+1)}(x^t, y^t) - D_{T(t)}(x^t, y^t))], \quad (12)$$

where the first difference (inside the first square brackets) measures technical efficiency change of the proportional distance function between periods t and $t+1$, while the arithmetic mean of the two last differences (inside the second square brackets) captures technological change. The latter represents the shift of technology between the two periods, evaluated at the input-output levels at $t+1$ and at the input-output levels realized at t.

Recapitulating the main differences between Malmquist and Luenberger productivity indices, one can distinguish between differences related to: (i) the choice of distance function (Shephard vs. proportional distance function); (ii) the economic motivation (cost or revenue optimization vs. profit maximization); and (iii) the nature of the index definition (ratio vs. difference) and the resulting decomposition (multiplicative vs. additive).

III. LUENBERGER AND MALMQUIST INDICES: A THEORETICAL COMPARISON

We start with a first proposition comparing proportional distance functions and, in addition, establishing relationships between, on the one hand, input and output Debreu–Farrell measures and, on the other hand, the family of proportional distance functions. This forms the basis for comparing the Luenberger and Malmquist productivity indices in a second proposition. Again, we determine both the relations among these recent Luenberger productivity indices and we relate the new approach to the more traditional Malmquist productivity indices.

Proposition 1: *Assume that in each time period t the production set T(t) satisfies (T.1)–(T.5).*

1) $D_{T(t)}(x^t, y^t) \leq \min\left\{D^i_{T(t)}(x^t, y^t), D^o_{T(t)}(x^t, y^t)\right\}$.

2) *Under constant returns to scale*,

$$a)\ E^i_{T(t)}(x^t, y^t) = \frac{1 - D_{T(t)}(x^t, y^t)}{1 + D_{T(t)}(x^t, y^t)} \text{ and } E^o_{T(t)}(x^t, y^t) = \frac{1 + D_{T(t)}(x^t, y^t)}{1 - D_{T(t)}(x^t, y^t)},$$

$$b)\ D_{T(t)}(x^t, y^t) = \frac{1 - E^i_{T(t)}(x^t, y^t)}{1 + E^i_{T(t)}(x^t, y^t)} = \frac{D^i_{T(t)}(x^t, y^t)}{2 - D^i_{T(t)}(x^t, y^t)},$$

$$c)\ D_{T(t)}(x^t, y^t) = \frac{E^o_{T(t)}(x^t, y^t) - 1}{E^o_{T(t)}(x^t, y^t) + 1} = \frac{D^o_{T(t)}(x^t, y^t)}{2 + D^o_{T(t)}(x^t, y^t)}.$$

Proof: 1) For simplicity, we denote $(\bar{x}^t, \bar{y}^t) = ((1 - D_{T(t)}(x^t, y^t)).x^t, (1 + D_{T(t)}(x^t, y^t)).y^t)$. Let $S(\bar{x}^t, \bar{y}^t) = \left\{(u^t, v^t) \in R^{n+p}_+; (-u^t, v^t) \leq (-\bar{x}^t, \bar{y}^t)\right\}$. It is immediate that $\max\{\delta \geq 0;\ ((1-\delta)x^t,\ (1+\delta)y^t) \in S(\bar{x}^t,\ \bar{y}^t)\} = D_{T(t)}(x^t,\ y^t)$. Moreover, it is trivial to show that:

$$\begin{aligned}\max\left\{\delta \geq 0; ((1-\delta)x^t, y^t) \in S(\bar{x}^t, \bar{y}^t)\right\} &= \max\left\{\delta \geq 0; (x^t, (1+\delta)y^t) \in S(\bar{x}^t, \bar{y}^t)\right\} \\ &= D_{T(t)}(x^t, y^t).\end{aligned}$$

However, since the strong disposability assumption holds: $S(\bar{x}^t,\ \bar{y}^t) \subset T(t)$. Hence, we deduce that $D_{T(t)}(x^t, y^t) = \max\left\{\delta \geq 0; ((1-\delta)x^t, y^t) \in S(\bar{x}^t, \bar{y}^t)\right\} \leq D^i_{T(t)}(x^t, y^t)$. Moreover, $D_{T(t)}(x^t, y^t) = \max\left\{\delta \geq 0; (x^t, (1+\delta)y^t) \in S(\bar{x}^t, \bar{y}^t)\right\} \leq D^o_{T(t)}(x^t, y^t)$, and the result follows.

2) We first prove *a*). Since returns to scale are constant, $\forall (x^t,\ y^t) \in T(t)$, $\lambda \geq 0 \Rightarrow (\lambda x^t,\ \lambda y^t) \in T(t)$. This implies that $(\lambda E^i_{T(t)}(x^t, y^t).x^t, \lambda y^t) \in T(t)$. Since the projected vector $(E^i_T(x^t, y^t).x^t, y^t)$ is a frontier point in $T(t)$ that achieves the Debreu–Farrell efficiency measure and since $((1 - D_{T(t)}(x^t, y^t)).x^t, (1 + D_{T(t)}(x^t, y^t)).y^t)$ achieves the proportional distance function, we need to find some $\lambda \geq 0$ that satisfies the relationship $(\lambda E^i_{T(t)}(x^t, y^t).x^t, \lambda y^t) = ((1 - D_T(x^t, y^t)).x^t, (1 + D_T(x^t, y^t)).y^t)$. Then, we deduce the following equalities: $\lambda E^i_{T(t)}(x^t, y^t) = 1 - D_{T(t)}(x^t, y^t)$ and $\lambda = 1 + D_{T(t)}(x^t, y^t)$. Dividing the first equation with the second yields: $E^i_{T(t)}(x^t, y^t) = \frac{1 - D_{T(t)}(x^t,\ y^t)}{1 + D_{T(t)}(x^t,\ y^t)}$. Since $E^o_{T(t)}(x^t, y^t) = [E^i_{T(t)}(x^t, y^t)]^{-1}$, the second relation in part *a*) follows suit. Parts *b*) and *c*) are also immediate. ■

In words, the first part reads that the proportional distance function is smaller or equal to the minimum of its input- and output-oriented versions. This result is somehow similar to the results that have been earlier established between several types of radial efficiency measures (Färe, Grosskopf and Lovell, 1985). The second part establishes some hitherto unnoticed links between Debreu–Farrell measures and the proportional distance functions. More precisely, part *a*) relates input- and output-oriented radial efficiency measures to the proportional distance function; parts *b*) and *c*) establish relationships between the proportional distance function and, respectively, the input- and output-oriented radial efficiency measures and their similarly oriented proportional distance functions.

In the literature so far, the Malmquist productivity index has been related to both the more traditional Fisher and Törnqvist productivity indices (see Färe, Grosskopf and Roos, 1998, pp. 139–40, for details). In particular, it has been established that the Malmquist output-oriented productivity index (4) is approximately equal to the Fisher productivity index. Under constant returns to scale, translog distance functions and some other conditions, the same Malmquist output-oriented productivity index equals the Törnqvist productivity index.

Using Proposition 1, we now link the Luenberger productivity indices to the traditional Malmquist indices in the next proposition.

Proposition 2: *Assume that the production set T satisfies (T.1)–(T.5). Under a constant returns to scale assumption, one finds*:

1) *At the second order*:

a) $\log(M^i(x^t, y^t, x^{t+1}, y^{t+1})) \cong -2 \cdot L(x^t, y^t, x^{t+1}, y^{t+1})$,

b) $\log(M^o(x^t, y^t, x^{t+1}, y^{t+1})) \cong 2 \cdot L(x^t, y^t, x^{t+1}, y^{t+1})$.

2) *At the first order*:

a) $\log(M^i(x^t, y^t, x^{t+1}, y^{t+1})) \cong -L^i(x^t, y^t, x^{t+1}, y^{t+1})$,

b) $\log(M^o(x^t, y^t, x^{t+1}, y^{t+1})) \cong L^o(x^t, y^t, x^{t+1}, y^{t+1})$,

c) $L(x^t, y^t, x^{t+1}, y^{t+1}) \cong \frac{1}{2}\left[L^o(x^t, y^t, x^{t+1}, y^{t+1}) - L^i(x^t, y^t, x^{t+1}, y^{t+1})\right]$.

Proof: 1) Let us show *a*). By definition, we have the equality $M^i(x^t, y^t, x^{t+1}, y^{t+1}) = \left[\frac{E^i_{T(t)}(x^t, y^t)}{E^i_{T(t)}(x^{t+1}, y^{t+1})} \frac{E^i_{T(t+1)}(x^t, y^t)}{E^i_{T(t+1)}(x^{t+1}, y^{t+1})}\right]^{1/2}$. Since Proposition 1.2.*a* can be applied to any two time periods t and $t+1$, we know that:

$$M^i(x^t, y^t, x^{t+1}, y^{t+1}) = \left[\frac{1 - D_{T(t)}(x^t, y^t)}{1 + D_{T(t)}(x^t, y^t)} \cdot \frac{1 + D_{T(t)}(x^{t+1}, y^{t+1})}{1 - D_{T(t)}(x^{t+1}, y^{t+1})} \cdot \frac{1 - D_{T(t+1)}(x^t, y^t)}{1 + D_{T(t+1)}(x^t, y^t)} \cdot \frac{1 + D_{T(t+1)}(x^{t+1}, y^{t+1})}{1 - D_{T(t+1)}(x^{t+1}, y^{t+1})}\right]^{1/2}.$$

Taking the logarithm on both sides and making a second order approximation yields:

$$\log(1 + D_{T(t)}(x^t, y^t)) \cong D_{T(t)}(x^t, y^t) - \left[D_{T(t)}(x^t, y^t)\right]^2, \text{and}$$

$$\log(1 - D_{T(t)}(x^t, y^t)) \cong -D_{T(t)}(x^t, y^t) - \left[D_{T(t)}(x^t, y^t)\right]^2.$$

Substituting in the above equation, one finds: log $(M^i(x^t, y^t, x^{t+1}, y^{t+1}))$ $\cong -2 \cdot L(x^t, y^t, x^{t+1}, y^{t+1})$. Part *b*) is obtained in a similar way.

2) Properties *a*) and *b*) are immediate using the fact that at the first order log $(1 + D_{T(t)}(x^t, y^t)) \cong D_{T(t)}(x^t, y^t)$ and log $(1 - D_{T(t)}(x^t, y^t)) \cong -D_{T(t)}(x^t, y^t)$. Part *c*) follows from 1.*a*) and 1.*b*) making a first order approximation. ■

Proposition 2 shows that the logarithm of the input Malmquist productivity index (to obtain percentage changes in productivity growth) is twice a linear approximation of minus the Luenberger productivity index. Clearly, since the input-oriented Debreu–Farrell measure involves only a modification of the input factors, it overestimates technical inefficiencies. As a consequence, one expects the Malmquist productivity index to overestimate productivity changes. Furthermore, this same proposition also links input- and output-oriented Malmquist productivity indices to their Luenberger counterparts. Last, but not least, this proposition compares the family of Luenberger productivity indices among themselves.

These results are of both theoretical and empirical interest. Theoretically, it is important to establish relations among productivity indices based on different types of distance functions. From an applied perspective, researchers should be able to anticipate differences in magnitudes of empirical results linked to different families of productivity indices. Section IV provides an empirical illustration of the main methodological result linking the input Malmquist productivity index to the Luenberger productivity index.

IV. LUENBERGER AND MALMQUIST INDICES: AN EMPIRICAL COMPARISON

Macroeconomic productivity gains have been estimated for 20 OECD countries over the 1974–97 period adopting alternatively the Luenberger and the input-oriented Malmquist productivity index. The macroeconomic production technology is defined as a simple single output that is produced by two inputs: capital and labour. This single output is gross domestic product (GDP) evaluated at 1990 prices. Capital stock estimates have been constructed using a perpetual inventory model with a delay of 15 years. The labour factor is defined as the number of employees. The nominal series (GDP and capital) at 1990 national prices are expressed in PPP US dollars. All data are taken from the OECD Main Economic Indicators.

The results obtained for the Luenberger and input Malmquist productivity indices are presented in Table 1. Looking at the Luenberger index, our sample reveals a moderate average total factor productivity growth rate of 0.39% per year, mainly explained by the technological change component (0.34%). The input Malmquist index, by contrast, detects a more impressive average total factor productivity growth rate of 0.78% over this period, again mainly due to technological changes that have also doubled in size (0.68%). Since most organizations contributing to the construction of the macroeconomic production possibility set operate in more or less competitive conditions, one may assume that they are able to adjust both inputs and output. Hence, the Luenberger productivity index comes closest to characterizing true factor productivity growth. The Malmquist index then presents an upwardly biased estimate.

Notice that all results reported have been obtained by using the proper productivity index computations. To illustrate the relations established in Proposition 2, we also briefly report on the approximation of the Luenberger index by minus the log of the input Malmquist index divided by two. Using this approximation for the whole sample, the Mean Absolute Deviation and Root Mean Square Error between true and approximate Luenberger indices amounts to 5.78E–05 and 10.74E–05, respectively. Clearly, this approximation is very satisfactory.

Looking at individual countries, results appear rather heterogeneous. On the one hand, Norway and Finland have the highest productivity growth rates (1.13%) — mainly due to technical progress. On the other hand, Canada and New Zealand experience negative productivity growth rates (–0.47% per year). The ranking of countries is preserved under the Malmquist index in terms of total factor productivity, efficiency and technical progress scores. But, again, the Malmquist growth rates are nearly twice the rates estimated by the Luenberger index and its components.

TABLE 1

Average annual growth rates (%) of TFP, efficiency and technological changes for Luenberger and Malmquist indices (1974–97)

	TFP		*Efficiency changes*		*Technological changes*	
	Luenberger	*Malmquist*	*Luenberger*	*Malmquist*	*Luenberger*	*Malmquist*
Australia	0.33	0.68	0.10	0.21	0.23	0.47
Austria	0.15	0.31	−0.27	−0.56	0.42	0.86
Belgium	0.45	0.89	0.13	0.27	0.31	0.63
Canada	−0.62	−1.24	−0.56	−1.12	−0.07	−0.13
Denmark	0.81	1.62	0.23	0.46	0.58	1.16
Finland	1.13	2.25	0.16	0.31	0.97	1.94
France	0.31	0.63	−0.09	−0.18	0.40	0.81
Germany	0.70	1.41	0.14	0.28	0.56	1.13
Iceland	0.37	0.75	−0.07	−0.14	0.44	0.89
Ireland	0.55	1.10	0.64	1.29	−0.09	−0.19
Italy	0.47	0.95	0.14	0.28	0.33	0.67
Japan	0.25	0.50	−0.19	−0.39	0.43	0.89
Luxembourg	0.69	1.40	0.25	0.50	0.45	0.90
Netherlands	0.41	0.83	−0.03	−0.07	0.45	0.90
New Zealand	−0.26	−0.52	−0.14	−0.29	−0.11	−0.22
Norway	1.13	2.27	0.07	0.14	1.06	2.13
Spain	0.10	0.20	0.15	0.30	−0.05	−0.10
Sweden	0.51	1.02	0.00	−0.01	0.52	1.03
United Kingdom	0.18	0.36	0.29	0.58	−0.11	−0.22
United States	0.11	0.23	0.04	0.08	0.07	0.15
Average	**0.39**	**0.78**	**0.05**	**0.10**	**0.34**	**0.68**

Focusing in particular on the French economy as an example in case, Figure 1 underlines this country's inefficiency under alternative measures for the whole period. A positive value for the proportional distance function indicates that the country could have both increased its output and decreased its inputs by this percentage with respect to the benchmark constructed from the most productive of these 20 countries. E.g., in 1997 France could have been able to simultaneously increase its output and decrease its inputs by 9.5% to catch up the production frontier (defined by Denmark, Finland, Ireland, Norway and Sweden). Adopting a Debreu–Farrell inefficiency measure (i.e., one minus the Debreu–Farrell efficiency measure), by contrast, France could have increased its output by 17.2% with constant inputs or decreased its inputs by this same percentage with a constant level of output. Remember that we use the same production frontier (defined by the same above countries) for each index. Figure 2 completes these findings for the logarithmic value of total factor productivity. As indicated previously, the two curves follow the same trend, but the Malmquist index variations are twice as large as the variations in the Luenberger index.

V. CONCLUDING REMARKS

This contribution has shown that the Malmquist productivity index, that involves a modification of either inputs or outputs, overestimates the productivity change compared to the Luenberger index, which looks for simultaneous input contractions and output expansions. Under a constant return to scale technology, the growth rates given by the former are nearly twice the rates estimated by the latter. Since the assumption of

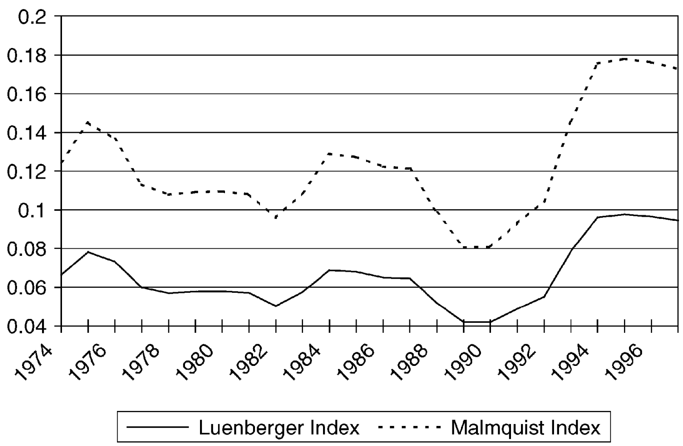

Fig. 1. Inefficiency scores (%) for France over the 1974–97 period

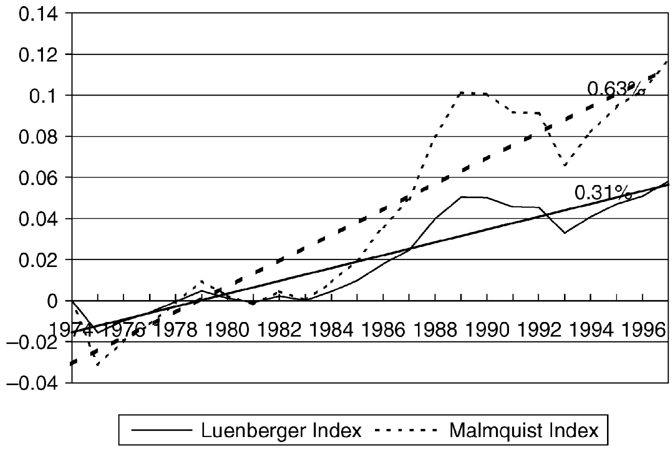

Fig. 2. Total Factor Productivity indices for France over the 1974–97 period

profit maximization underlying the Luenberger index is more generally valid, this key result implies that the older ways of measuring productivity may well be in error by a factor of two.

Putting both indices in perspective, Malmquist and Luenberger productivity indices belong to, respectively, the ratio and the difference approaches to index numbers. A major difference is that the former index is based on traditional distance functions that are multiplicative in nature, while the latter index is based on proportional distance functions that have an additive structure. Difference-based indices are, relatively speaking, new in the field of economics, but they clearly may become more important in the future.

An empirical application of these measures on a sample of 20 OECD economies illustrates this relationship. Applied researchers should beware of misinterpreting differences in magnitudes of empirical results that are a simple consequence of employing different families of productivity indices with different behavioural assumptions.

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