Economic Theory **23**, 925–939 (2004)
DOI: 10.1007/s00199-003-0403-2

Economic Theory

Exposita Notes

A Luenberger-Hicks-Moorsteen productivity indicator: its relation to the Hicks-Moorsteen productivity index and the Luenberger productivity indicator*

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Received: May 2, 2002; revised version: May 19, 2003

Summary. Introducing a new difference-based Luenberger-Hicks-Moorsteen productivity indicator, this contribution establishes theoretically its relations with some existing ratio- and difference-based productivity indexes and indicators. The first main result is an approximation proposition stating that the logarithm of the Hicks-Moorsteen productivity index is about equal to the Luenberger-Hicks-Moorsteen productivity indicator. Secondly, we also establish the specific conditions under which the Luenberger-Hicks-Moorsteen indicator equals the recently introduced Luenberger indicator and compare these to the conditions governing the relations between ratio-based Hicks-Moorsteen and Malmquist indices.

Keywords and Phrases: Malmquist productivity index, Hicks-Moorsteen productivity index, Luenberger productivity indicator, Luenberger-Hicks-Moorsteen productivity indicator.

JEL Classification Numbers: C43, D21, D24.

1 Introduction

Total factor productivity (TFP) growth, as estimated by the traditional Solow residual, yields an index number reflecting shifts in technology resulting from output

* We are grateful for the most constructive remarks of a referee. Obviously, the authors remain solely responsible for any residual shortcomings.

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growth left unexplained by input growth (Hulten (2001)). Awareness has grown that ignoring inefficiency biases TFP measures. Nishimizu and Page (1981) were the first to decompose TFP into technical change and technical efficiency change using parametric production frontiers. Caves, Christensen and Diewert (1982) analyse Malmquist input, output and productivity indexes in discrete time using distance functions as general technology representations. Since these indexes require a precise knowledge of technology, they develop an empirical estimation strategy by relating Malmquist and Törnqvist productivity indexes, the latter needing no exact knowledge of technology but depending on both prices and quantities. Färe et al. (1995) integrate the two-part decomposition of TFP of Nishimizu and Page (1981) into the Malmquist index and make it computationally tractable by exploiting the relation between distance functions and radial efficiency measures. Bjurek (1996) defines an alternative Malmquist TFP (or Hicks-Moorsteen) index, as a ratio of Malmquist output and input indices.

Luenberger (1992) generalises existing distance functions by introducing the shortage function, which accounts for both input contractions and output improvements, and establishes duality between this shortage (or directional distance) function and the profit function (see also Chambers, Chung and Färe, 1998). A Luenberger productivity indicator is defined by Chambers (2002) as a difference-based index of directional distance functions.[1] Extending Chambers (1998), we introduce a new difference-based variation on the Hicks-Moorsteen productivity index, which is labelled the Luenberger-Hicks-Moorsteen indicator.

This paper first introduces a series of ratio-based and difference-based primal productivity indices and indicators in a discrete time framework and then proves two main theoretical results. First, an approximation result shows that the Luenberger-Hicks-Moorsteen indicator is about equal to the logarithm of the Hicks-Moorsteen index, its ratio-based counterpart. This outcome complements recent approximation results linking the Luenberger indicator to the Malmquist index, a ratio-based version of the former. In particular, Boussemart et al. (2003) prove that the logarithm of the input Malmquist productivity index is twice a linear approximation of minus the Luenberger productivity indicator. In this way all currently known primal productivity indices and indicators are related to one another.

Second, the necessary and sufficient conditions to obtain equality between difference-based Luenberger-Hicks-Moorsteen and Luenberger output (or input) oriented productivity indicators are determined. Both coincide under two properties: (i) a new property of inverse translation homotheticity of technology in the direction of g (see also Fukuyama, 2002); and (ii) graph translation homotheticity in the direction of g (Chambers, 2002). These two conditions are the arithmetic counterparts of the ones linking the ratio-based Malmquist and Hicks-Moorsteen productivity indexes (Färe, Grosskopf and Roos, 1996): (i) inverse homotheticity of technology; and (ii) constant returns to scale (CRS). Similar to Färe, Grosskopf and Roos (1996), we conclude that Luenberger-Hicks-Moorsteen and Luenberger

[1] Following Diewert (1998), "indicators" denote productivity measures based on differences, while "indexes" indicate productivity measures defined as ratios. A systematic comparison of ratio and difference approaches to index number theory from both a test and an economic perspective is found in Chambers (1998, 2002) and Diewert (1998), among others.

output- and input-oriented productivity indicators will in general differ, since the conditions needed for their equality are strong and unlikely to be met in empirical work.

This contribution is structured as follows. In Section 2 the assumptions on technology and the definitions of the various distance functions and productivity indices are developed. The next section states the two main theoretical propositions of this contribution.

2 Definitions of technology, distance functions and productivity indices and indicators

2.1 Technology and distance functions

Production technology transforms inputs $x = (x_1, ..., x_n) \in R_+^n$ into outputs $y = (y_1, ..., y_p) \in R_+^p$. For each time period t, the production possibility set $T(t)$ summarises the set of all feasible input and output vectors and is defined as follows:

$$T(t) = \left\{(x^t, y^t) \in R_+^{n+p}; \quad x^t \text{ can produce } y^t\right\} . \tag{1}$$

Throughout the paper technology satisfies the following conventional assumptions: (T.1) $(0,0) \in T(t), (0, y^t) \in T(t) \Rightarrow y^t = 0$ i.e., no free lunch; (T.2) the set $A(x^t) = \{(u^t, y^t) \in T(t); u^t \leq x^t\}$ of dominating observations is bounded $\forall x^t \in R_+^n$, i.e., infinite outputs are not allowed with a finite input vector; (T.3) $T(t)$ is closed; and (T.4) $\forall (x^t, y^t) \in T(t), (x^t, -y^t) \leq (u^t, -v^t) \Rightarrow (u^t, v^t) \in T(t)$, i.e., fewer outputs can always be produced with more inputs, and inversely (strong disposal of inputs and outputs). On one occasion, stronger assumptions (specifically, convexity) are needed.

Efficiency is estimated relative to production frontiers using distance or gauge functions. The directional distance function $D_{T(t)}\left(.,.; g^t\right) : T(t) \to R$ involving a simultaneous input and output variation in the direction of a pre-assigned vector $g^t = (g_i^t, g_o^t) \in R_+^{n+p}$ is defined as:

$$D_{T(t)}\left(x^t, y^t; g^t\right) = \max_{\delta}\left\{\delta \geq 0; \left(x^t - \delta g_i^t, y^t + \delta g_o^t\right) \in T(t)\right\} . \tag{2}$$

This directional distance function (Chambers, Färe and Grosskopf, 1996) is a special case of the shortage function (Luenberger, 1992). In the remainder, we denote by $D_{T(b)}\left(x^a, y^a; g^a\right) = \max_{\delta}\{\delta \in R; (x^a - \delta g_i^a, y^a + \delta g_o^a) \in T(b)\}$ the time-related version of this directional distance function, where $(a, b) \in \{t, t+1\} \times \{t, t+1\}$. Note that this function is defined using a general directional vector g, while we consider the special case: $g_i^t = x^t$ and $g_o^t = y^t$. The latter is known as the Farrell proportional distance function (Briec, 1997), a generalization of the Farrell measure.[2] From Chambers, Färe and Grosskopf (1996), the Farrell input measure is defined as one minus the input-oriented proportional distance function: $E_{T(t)}^i(x^t, y^t) = 1 - D_{T(t)}(x^t, y^t; (x^t, 0))$. Similarly, the Farrell output efficiency measure is defined as one plus the output-oriented proportional distance function: $E_{T(t)}^o(x^t, y^t) = 1 + D_{T(t)}(x^t, y^t; (0, y^t))$.

[2] Axiomatic properties of this function are studied in Briec (1997) and Chambers, Chung and Färe (1998).

2.2 Productivity indices and indicators

2.2.1 Ratio-based productivity indices

Having defined the necessary components, we can define the productivity indices and indicators. Using the input Farrell measures, the input-oriented Malmquist productivity index $M^i((x^t,y^t),(x^{t+1},y^{t+1}))$ is defined as follows:

$$M^i\left(\left(x^t,y^t\right),\left(x^{t+1},y^{t+1}\right)\right) \\ = \left[\frac{E^i_{T(t)}\left(x^t,y^t\right)}{E^i_{T(t)}\left(x^{t+1},y^{t+1}\right)}\frac{E^i_{T(t+1)}\left(x^t,y^t\right)}{E^i_{T(t+1)}\left(x^{t+1},y^{t+1}\right)}\right]^{1/2}. \tag{3}$$

To avoid an arbitrary selection among base years, a geometric mean of period t (first ratio) and period $t+1$ (second ratio) Malmquist indices is taken. Productivity growth (decline) is indicated by values below (above) unity. Similarly, a Malmquist output productivity index is defined as follows:

$$M^o\left(\left(x^t,y^t\right),\left(x^{t+1},y^{t+1}\right)\right) \\ = \left[\frac{E^o_{T(t)}\left(x^t,y^t\right)}{E^o_{T(t)}\left(x^{t+1},y^{t+1}\right)}\frac{E^o_{T(t+1)}\left(x^t,y^t\right)}{E^o_{T(t+1)}\left(x^{t+1},y^{t+1}\right)}\right]^{1/2}. \tag{4}$$

Since $E^o_{T(t)}\left(x^t,y^t\right)=\left[E^i_{T(t)}\left(x^t,y^t\right)\right]^{-1}$ under CRS, evidently:

$$M^o\left(\left(x^t,y^t\right),\left(x^{t+1},y^{t+1}\right)\right)=\left[M^i\left(\left(x^t,y^t\right),\left(x^{t+1},y^{t+1}\right)\right)\right]^{-1}.$$

Following Bjurek (1996), a Hicks-Moorsteen productivity (or Malmquist TFP) index with base period t is defined as the ratio of a Malmquist output quantity index at base period t and a Malmquist input quantity index at base period t:

$$HM_{T(t)}\left(x^t,y^t,x^{t+1},y^{t+1}\right)=\frac{E^o_{T(t)}\left(x^t,y^t\right)\Big/E^o_{T(t)}\left(x^t,y^{t+1}\right)}{E^i_{T(t)}\left(x^t,y^t\right)\Big/E^i_{T(t)}\left(x^{t+1},y^t\right)} \\ \equiv\frac{MO_{T(t)}\left(x^t,y^t,y^{t+1}\right)}{MI_{T(t)}\left(x^t,x^{t+1},y^t\right)}. \tag{5}$$

When the Hicks-Moorsteen productivity index is larger (smaller) than unity, it indicates productivity gain (loss).[3] A base period $t+1$ Hicks-Moorsteen productivity index is defined as follows:

$$HM_{T(t+1)}\left(x^t,y^t,x^{t+1},y^{t+1}\right)=\frac{E^o_{T(t+1)}\left(x^{t+1},y^t\right)\Big/E^o_{T(t+1)}\left(x^{t+1},y^{t+1}\right)}{E^i_{T(t+1)}\left(x^t,y^{t+1}\right)\Big/E^i_{T(t+1)}\left(x^{t+1},y^{t+1}\right)} \\ \equiv\frac{MO_{T(t+1)}\left(x^{t+1},y^{t+1},y^t\right)}{MI_{T(t+1)}\left(x^t,x^{t+1},y^{t+1}\right)}. \tag{6}$$

[3] The Malmquist input quantity index is discussed in Chambers, Färe and Grosskopf (1994). The same authors also discuss the Deaton index, a variation on the Malmquist output quantity index.

A geometric mean of these two Hicks-Moorsteen productivity indexes is (Bjurek, 1996, p. 310):

$$HM_{T(t),T(t+1)}\left(x^{t},y^{t},x^{t+1},y^{t+1}\right) = \left[HM_{T(t)}\left(x^{t},y^{t},x^{t+1},y^{t+1}\right).HM_{T(t+1)}\left(x^{t},y^{t},x^{t+1},y^{t+1}\right)\right]^{1/2}. \tag{7}$$

2.2.2 Difference-based productivity indicators

Chambers (2002) defines the Luenberger productivity indicator $L((x^t,y^t),(x^{t+1},y^{t+1});g^t,g^{t+1})$ in the general case of directional distance functions as follows:

$$\begin{aligned} &L\left(\left(x^{t},y^{t}\right),\left(x^{t+1},y^{t+1}\right);g^{t},g^{t+1}\right) \\ &= \frac{1}{2}\Big[\left(D_{T(t)}\left(x^{t},y^{t};g^{t}\right) - D_{T(t)}\left(x^{t+1},y^{t+1};g^{t+1}\right)\right) \\ &+ \left(D_{T(t+1)}\left(x^{t},y^{t};g^{t}\right) - D_{T(t+1)}\left(x^{t+1},y^{t+1}g^{t+1}\right)\right)\Big]. \end{aligned} \tag{8}$$

When $g^t = (x^t,y^t)$ and $g^{t+1} = \left(x^{t+1},y^{t+1}\right)$, then one obtains a proportional indicator, as mentioned in Chambers, Färe and Grosskopf (1996). In an effort to avoid an arbitrary choice of base years, an arithmetic mean of a difference-based Luenberger productivity indicator in base year t (first difference) and $t+1$ (second difference) has been taken. Productivity growth (decline) is indicated by positive (negative) values.

It is equally possible to define input- and output-oriented versions of this Luenberger productivity indicator based on input respectively output directional distance functions. The input Luenberger productivity indicator is defined as follows:

$$\begin{aligned} &L^{i}\left(\left(x^{t},y^{t}\right),\left(x^{t+1},y^{t+1}\right);g_i^{t},g_i^{t+1}\right) \\ &= L\left(\left(x^{t},y^{t}\right),\left(x^{t+1},y^{t+1}\right);\left(g_i^{t},0\right),\left(g_i^{t+1},0\right)\right), \end{aligned} \tag{9}$$

while the output Luenberger productivity indicator can be defined as:

$$\begin{aligned} &L^{o}\left(\left(x^{t},y^{t}\right),\left(x^{t+1},y^{t+1}\right);g_o^{t},g_o^{t+1}\right) \\ &= L\left(\left(x^{t},y^{t}\right),\left(x^{t+1},y^{t+1}\right);\left(0,g_o^{t}\right),\left(0,g_o^{t+1}\right)\right). \end{aligned} \tag{10}$$

These are difference-based indicators of the similarly oriented ratio-based Malmquist indices.

Extending some basic elements developed in Chambers (1998, 2002), a Luenberger-Hicks-Moorsteen indicator with base period t is defined as the difference between a Luenberger output quantity indicator and a Luenberger input quantity indicator:[4]

[4] Actually, Chambers (1998) defines these difference-based Luenberger input and output indicators using a special case of the shortage (directional distance) function known as the translation function. Furthermore, we slightly modify the Chambers (1998, 2002) definition of the Luenberger input quantity indicator to ensure that improvements of the Luenberger-Hicks-Moorsteen indicator are positively signed.

$$\begin{aligned}
&LHM_{T(t)}\left(x^{t+1}, y^{t+1}, x^t, y^t; g^t, g^{t+1}\right) \\
&= \left(D_{T(t)}\left(x^t, y^t; \left(0, g_o^t\right)\right) - D_{T(t)}\left(x^t, y^{t+1}; \left(0, g_o^{t+1}\right)\right)\right) \\
&\quad - \left(D_{T(t)}\left(x^{t+1}, y^t; \left(g_i^{t+1}, 0\right)\right) - D_{T(t)}\left(x^t, y^t; \left(g_i^t, 0\right)\right)\right) \\
&\equiv LO_{T(t)}\left(x^t, y^t, y^{t+1}; g_o^t, g_o^{t+1}\right) - LI_{T(t)}\left(x^t, x^{t+1}, y^t; g_i^t, g_i^{t+1}\right).
\end{aligned} \tag{11}$$

These Luenberger output quantity and input quantity indicators generalise the Malmquist output and input quantity indices defined in Chambers, Färe and Grosskopf (1994). A Luenberger-Hicks-Moorsteen productivity indicator larger (smaller) than zero indicates productivity gain (loss). A base period $t+1$ Luenberger-Hicks-Moorsteen indicator can be similarly defined:

$$\begin{aligned}
&LHM_{T(t+1)}\left(x^{t+1}, y^{t+1}, x^t, y^t; g^t, g^{t+1}\right) \\
&= \left(D_{T(t+1)}\left(x^{t+1}, y^t; \left(0, g_o^t\right)\right) - D_{T(t+1)}\left(x^{t+1}, y^{t+1}; \left(0, g_o^{t+1}\right)\right)\right) \\
&\quad - \left(D_{T(t+1)}\left(x^{t+1}, y^{t+1}; \left(g_i^{t+1}, 0\right)\right) - D_{T(t+1)}\left(x^t, y^{t+1}; \left(g_i^t, 0\right)\right)\right) \\
&\equiv LO_{T(t+1)}\left(x^{t+1}, y^{t+1}, y^t; g_o^t, g_o^{t+1}\right) - LI_{T(t+1)}\left(x^t, x^{t+1}, y^{t+1}; g_i^t, g_i^{t+1}\right).
\end{aligned} \tag{12}$$

An arithmetic mean of these two base periods Luenberger-Hicks-Moorsteen indicators is:

$$\begin{aligned}
&LHM_{T(t),T(t+1)}\left(x^t, y^t, x^{t+1}, y^{t+1}; g^t, g^{t+1}\right) \\
&= \frac{1}{2}\Big[LHM_{T(t)}\left(x^t, y^t, x^{t+1}, y^{t+1}; g^t, g^{t+1}\right) \\
&\quad + LHM_{T(t+1)}\left(x^t, y^t, x^{t+1}, y^{t+1}; g^t, g^{t+1}\right)\Big].
\end{aligned} \tag{13}$$

3 Productivity indices and indicators: some theoretical comparisons

We establish two main results.[5] The first links the Luenberger-Hicks-Moorsteen indicator and its ratio-based Hicks-Moorsteen counterpart. Second, we determine the exact conditions on technology that make the Luenberger-Hicks-Moorsteen indicator and the input-or output-oriented versions of the Luenberger productivity indicators coincide.

3.1 Hicks-Moorsteen index and Luenberger-Hicks-Moorsteen indicator

A linear approximation result for the Luenberger-Hicks-Moorsteen indicator and the Hicks-Moorsteen index is found in the next proposition.

[5] Färe, Grosskopf and Roos (1998) summarise in detail the relations between these primal discrete-time productivity indices and more traditional approaches. In particular, they show that the Malmquist productivity indexes are far more general than the Fisher and Törnqvist indexes, among others because they do not require knowledge of prices nor assume any form of optimising behaviour.

Proposition 1. *Assume that technology $T(t)$ satisfies (T.1)–(T.4). If at each time period $g^t = (x^t, y^t)$ and $g^{t+1} = \left(x^{t+1}, y^{t+1}\right)$, then at the first order:*

$$LHM_{T(t),T(t+1)}\left(x^t, y^t, x^{t+1}, y^{t+1}; g^t, g^{t+1}\right)$$
$$\approx \log\left(HM_{T(t),T(t+1)}\left(x^t, y^t, x^{t+1}, y^{t+1}\right)\right).$$

Proof. Since the equalities $E^i_{T(t)}\left(x^t, y^t\right) = 1 - D_{T(t)}\left(x^t, y^t; \left(x^t, 0\right)\right)$ and $E^o_{T(t)}$ $\left(x^t, y^t\right) = 1 + D_{T(t)}\left(x^t, y^t; \left(0, y^t\right)\right)$ work at each time period, at the first order we have

$$\log\left(E^i_{T(b)}\left(x^a, y^a\right)\right) \approx -D_{T(b)}\left(x^a, y^a; \left(x^a, 0\right)\right) \text{ and}$$
$$\log\left(E^o_{T(t)}\left(x^t, y^t\right)\right) \approx D_{T(b)}\left(x^a, y^a; \left(0, y^a\right)\right)$$
$$\text{for } (a, b) \in \{t, t+1\} \times \{t, t+1\}. \tag{a}$$

Now taking the logarithm of (5), we obtain at the first order:

$$\log\left(HM_{T(t)}\left(x^t, y^t, x^{t+1}, y^{t+1}\right)\right)$$
$$\approx LHM_{T(t)}\left(x^t, y^t, x^{t+1}, y^{t+1}; g^t, g^{t+1}\right). \tag{b}$$

Similarly taking the logarithm of (6), we obtain:

$$\log\left(HM_{T(t+1)}\left(x^t, y^t, x^{t+1}, y^{t+1}\right)\right)$$
$$\approx LHM_{T(t+1)}\left(x^t, y^t, x^{t+1}, y^{t+1}; g^t, g^{t+1}\right) \tag{c}$$

and we immediately deduce the result. □

3.2 Graph and inverse translation homotheticity: definitions and intermediate results

In Subsection III.3, we study the relations between Luenberger input (output) and Luenberger-Hick-Moorsteen productivity indicators to duplicate the Färe, Grosskopf and Roos (1996) results for the ratio-based counterpart indices. This subsection develops intermediary definitions and propositions using the notion of graph translation homotheticity (Chambers, 2002). For notational simplicity, assume that $g^t = g$ at each time period t.

Definition 1. *Assume that technology $T(t)$ satisfies (T.1)–(T.4). At each time period t we say that T(t) exhibits graph translation homotheticity in the direction of g, if for any scalarδ:*

$$\left(x^t, y^t\right) \in T(t) \textit{ and } \left(x^t, y^t\right) + \delta g \geq 0 \Rightarrow \left(x^t, y^t\right) + \delta g \in T(t).$$

Remark that the condition in the above definition reduces to postulating that $T(t) = (T(t) + \Delta(g)) \cap R^{n+p}_+$, where $\Delta(g) = \{\delta g : \delta \in R\}$. We show below that this property can be characterised using the input- and output-oriented directional distance functions.

Proposition 2. *Assume that technology $T(t)$ satisfies (T.1)–(T.4). At each time period t, $T(t)$ satisfies graph translation homotheticity in the direction of g if and only if:*

$$D_{T(t)}\left(x^t,y^t;(g_i,0)\right)=D_{T(t)}\left(x^t,y^t;(0,g_o)\right)=2D_{T(t)}\left(x^t,y^t;(g_i,g_o)\right).$$

Proof. Assume that technology satisfies graph translation homotheticity. Consider the point:

$$\begin{aligned}\left(\bar{x}^t,\bar{y}^t\right)&=\left(x^t,y^t\right)-D_{T(t)}\left(x^t,y^t;(g_i,0)\right).(g_i,0)\\&=\left(x^t-D_{T(t)}\left(x^t,y^t;(g_i,0)\right).g_i,y^t\right)\end{aligned}\tag{a}$$

Now, let $\eta=D_{T(t)}\left(x^t,y^t;(g_i,0)\right)$, since $\eta\geq 0$ we have:

$$\begin{aligned}\left(\hat{x}^t,\hat{y}^t\right)&=\left(\bar{x}^t,\bar{y}^t\right)+D_{T(t)}\left(x^t,y^t;(g_i,0).g\right)\\&=\left(x^t,y^t+D_{T(t)}\left(x^t,y^t;(g_i,0).g_o\right)\right.\end{aligned}\tag{b}$$

Since technology exhibits graph translation homotheticity in the direction of g it is easy to show that $int\,(T(t))=(int\,(T(t))+\Delta(g))\cap R_+^{n+p}$ where int stands for the interior. Therefore, since from (a), $(\bar{x}^t,\bar{y}^t)$ belongs to the boundary the point $(\hat{x}^t,\hat{y}^t)$ belongs to the boundary of $T(t)$. Consequently,

$$\left(x^t,y^t+D_{T(t)}\left(x^t,y^t;(g_i,0)\right).g_o\right)=\left(x^t,y^t+D_{T(t)}\left(x^t,y^t;(0,g_o)\right).g_o\right).$$

As a consequence, $D_{T(t)}\left(x^t,y^t;(g_i,0)\right)=D_{T(t)}\left(x^t,y^t;(0,g_o)\right)$ and the first part of the equality is proven.

Now, let the point $(\tilde{x}^t,\tilde{y}^t)=\frac{1}{2}(\bar{x}^t,\bar{y}^t)+\frac{1}{2}(\hat{x}^t,\hat{y}^t)$. Since $(\hat{x}^t,\hat{y}^t)$ is the translation of $(\bar{x}^t,\bar{y}^t)$ in the direction of g, $(\tilde{x}^t,\tilde{y}^t)$ is also a translation of $(\bar{x}^t,\bar{y}^t)$ in the direction of g. Thus, since $T(t)$ satisfies graph translation homotheticity $(\tilde{x}^t,\tilde{y}^t)\in T(t)$. Moreover:

$$\begin{aligned}\left(\tilde{x}^t,\tilde{y}^t\right)&=\frac{1}{2}\left((x^t,y^t)-D_{T(t)}\left(x^t,y^t;(g_i,0)\right).(g_i,0)\right)\\&=\frac{1}{2}\left(\left(x^t,y^t\right)+D_{T(t)}\left(x^t,y^t;(0,g_o)\right).(0,g_o)\right)\end{aligned}\tag{c}$$

Since using the arguments above it can be proven that $(\tilde{x}^t,\tilde{y}^t)$ lies on the boundary of $T(t)$, we have $(\tilde{x}^t,\tilde{y}^t)=(x^t,y^t)+D_{T(t)}\left(x^t,y^t;(g_i,g_o)\right).(-g_i,g_o)$. It follows that we have

$$D_{T(t)}\left(x^t,y^t;(g_i,g_o)\right)=\frac{1}{2}D_{T(t)}\left(x^t,y^t;(g_i,0)\right)=\frac{1}{2}D_{T(t)}\left(x^t,y^t;(0,g_o)\right).$$

To show the converse, assume that $(x^t,y^t)\in T(t)$ and $(x^t,y^t)+\eta.g\geq 0$. Assume that $\eta\geq 0$ and let us consider the point $\left(x_+^t,y_+^t\right)=(x^t+\eta.g_i,y^t)\geq 0$. Immediately, we deduce that: $D_{T(t)}\left(x_+^t,y_+^t;(g_i,0)\right)\geq\eta$. Consequently, $D_{T(t)}\left(x_+^t,y_+^t;(0,g_o)\right)\geq\eta$ and since $(x^t,y^t)+\eta.g=\left(x_+^t,y_+^t\right)+\eta.(0,g_o)$ we deduce that this point belongs to $T(t)$. If $\eta\leq 0$, then the proof is similar defining the point $\left(x_-^t,y_-^t\right)=(x^t,y^t-\eta.g_o)\geq 0$. □

Furthermore, in addition to Proposition 2 the following characterization of a technology exhibiting graph translation homotheticity can be provided.

Proposition 3. *Assume that technology $T(t)$ satisfies (T.1)–(T.4) and $T(t)$ is convex:*

1) At each time period t, T (t) satisfies graph translation homotheticity in the direction of g if and only if $D_{T(t)}\left(x^t, y^t + \alpha g_o; (g_i, 0)\right) = D_{T(t)}\left(x^t, y^t; (g_i, 0)\right) - \alpha$.
2) At each time period t, T(t) satisfies graph translation homotheticity in the direction of g if and only if $D_{T(t)}\left(x^t + \alpha g_i^t, y^t; (0, g_o)\right) = D_{T(t)}\left(x^t, y^t; (0, g_o)\right) + \alpha$.

Proof. 1) The proof is obtained using the properties stated by Chambers, Chung and Färe (1998). If the condition in Proposition 2 holds:

$$\begin{aligned} D_{T(t)}\left(x^t, y^t + \alpha g_o; (g_i, 0)\right) &= D_{T(t)}\left(x^t, y^t + \alpha g_o; (0, g_o)\right) \\ &= D_{T(t)}\left(x^t, y^t; (0, g_o)\right) - \alpha \\ &= D_{T(t)}\left(x^t, y^t; (g_i, 0)\right) - \alpha \end{aligned}$$

Using the same equalities above we deduce that the condition in 2) implies the condition in 1). 2) is obtained symmetrically. □

Remark that the results above are to some extent analogous to those characterizing CRS for the traditional Shephard distance function, but now replacing homogeneity by translation homotheticity. The following corollary is immediate:

Corollary 1. *Assume that technology $T(t)$ satisfies (T.1)–(T.4). $T(t)$ exhibits graph translation homotheticity in the direction of g if and only if:*

$$L^i\left(\left(x^t, y^t\right), \left(x^{t+1}, y^{t+1}\right); g_i\right) = L^o\left(\left(x^t, y^t\right), \left(x^{t+1}, y^{t+1}\right); g_o\right)$$

Proof. See Theorem 7 in Chambers (2002).

This result is analogous to the well-known result that input-based and output-based Malmquist productivity indicators coincide under CRS, albeit inversely.

Before showing below how the above properties can help to specify the exact conditions under which Luenberger-Hicks-Moorsteen and Luenberger (input- or output-oriented) productivity indicators are identical, we introduce a new notion of translation inverse homothetic technologies (simultaneously proposed by Fukuyama, 2002). Translation homotheticity is a property of technologies developed by Chambers and Färe (1998).[6] The new property of inverse translation homotheticity extends the notion of inversely homothetic technologies, earlier developed by Färe and Primont (1995), to an arithmetic viewpoint.

Following Chambers and Färe (1998), technology is input translation homothetic if:

$$L\left(y^t\right) = H\left(y^t, g_o\right) g_o + L\left(\bar{y}\right) \text{ for } y^t \in R_+^p \tag{14}$$

[6] Färe and Li (2001) develop a test for translation homotheticity using non-parametric technologies.

where $\bar{y}$ is a fixed output vector. Remark that the original paper focuses on the case where this vector has all components equal to one. Moreover, $H\left(y^t,.\right)$ is assumed to be consistent with the properties of the directional distance function (see Chambers and Färe, 1998). By analogy with Färe and Primont (1995), output translation homotheticity is defined by:

$$P\left(x^t\right) = G\left(x^t, g_i\right) g_i + G\left(\bar{x}\right) \text{ for } x^t \in R_+^n \tag{15}$$

where $\bar{x}$ is a fixed input vector with all components being equal to one and $G\left(x^t,.\right)$ is consistent with the properties of the directional distance function. From Chambers and Färe (1998), the following properties can be stated:

Proposition 4. *Assume that technology $T(t)$ satisfies (T.1)–(T.4):*

1) The technology is input translation homothetic if and only if

$$D_{T(t)}\left(x^t, y^t; (g_i, 0)\right) = D_{T(t)}\left(x^t, \bar{y}; (g_i, 0)\right) - H\left(y^t, g_i\right).$$

2) The technology is output translation homothetic if and only if

$$D_{T(t)}\left(x^t, y^t; (0, g_o)\right) = D_{T(t)}\left(\bar{x}, y^t; (0, g_o)\right) + G\left(x^t, g_o\right).$$[7]

Proof. 1) is proven in Chambers and Färe (1998) replacing $\bar{y}$ by 1^p. A symmetrical way is used for proving 2). We have:

$$\begin{aligned}
&D_{T(t)}\left(x^t, y^t; (0, g_o)\right) \\
&= \sup\left\{\beta : y + \beta g_o \in P(x)\right\} \\
&= \sup\left\{\beta : y + \beta g_o \in G\left(x, g_o\right) + P\left(\bar{x}\right)\right\} \\
&= \sup\left\{\beta : y + g_o\left(\beta - G\left(x, g_o\right)\right) \in P\left(\bar{x}\right)\right\} \qquad \text{(a)} \\
&= \sup\left\{\beta - G\left(x, g_o\right) + G\left(x, g_o\right) : y + g_o\left(\beta - G\left(x, g_o\right)\right) \in P(\bar{x})\right\} \\
&= G\left(x, g_o\right) + \sup\left\{\delta : y + \delta g_o \in P(\bar{x})\right\} \\
&= D_{T(t)}\left(\bar{x}, y^t; (0, g_o)\right) + G\left(x^t, g_o\right).
\end{aligned}$$

Let us show the converse. We have:

$$\begin{aligned}
P\left(x^t\right) &= \left\{y^t : D_{T(t)}\left(\bar{x}, y^t, (0, g_o)\right) + G\left(x^t, g_o\right) \geq 0\right\} \\
&= \left\{y^t : D_{T(t)}\left(\bar{x}, y^t + G\left(x^t, g_o\right) g_o, (0, g_o)\right) \geq 0\right\} \qquad \text{(b)} \\
&= G\left(x^t, g_o\right) g_o + \left\{\tilde{y}^t : D_{T(t)}\left(\bar{x}, \tilde{y}^t, (0, g_o)\right) \geq 0\right\} \\
&= G\left(x^t, g_o\right) g_o + P\left(\bar{x}\right)
\end{aligned}$$

and the result is stated. □

Inspired by the result of Färe and Primont (1995), we now introduce the notion of a translation inverse homothetic technology.

[7] This definition is inspired by Färe and Primont (1995) and is slightly different from the one developed in Färe and Grosskopf (2000).

Definition 2. *A technology is translation inverse homothetic in the direction of g if and only if there exists an invertible function F such that:*

$$D_{T(t)}\left(x^t, y^t, (0, g_o)\right) = D_{T(t)}\left(\bar{x}, y^t; (0, g_o)\right) - F\left(D_{T(t)}\left(x^t, \bar{y}, (g_i, 0)\right)\right)$$

and

$$D_{T(t)}\left(x^t, y^t, (g_i, 0)\right) = D_{T(t)}\left(x^t, \bar{y}; (g_i, 0)\right) - F^{-1}\left(D_{T(t)}\left(\bar{x}, y^t, (0, g_o)\right)\right)$$

where F is a non-increasing and invertible function and $\bar{x}$ and $\bar{y}$ are two arbitrary vectors.

Fukuyama (2002) introduces an identical definition to verify the implications on the structure of separable cost and revenue functions.

Proposition 5. *Assume that technology T(t) satisfies (T.1)–(T.4). If the technology is such that $D_{T(t)}(x^t, y^t, (0, g_o)) = 0 \Leftrightarrow D_{T(t)}(x^t, y^t, (g_i, 0)) = 0$, then the technology T(t) is input and output translation homothetic*[8] *if and only if it is inversely translation homothetic.*

Proof. It is always true that:

$$D_{T(t)}\left(x^t - D_{T(t)}\left(x^t, y^t, (g_i, 0)\right).g_i, y^t; (g_i, 0)\right) = 0 \tag{a}$$

Thus, from Proposition 4.1, we obtain:

$$\begin{aligned} &D_{T(t)}\left(x^t - D_{T(t)}\left(x^t, y^t, (g_i, 0)\right).g_i, y^t; (0, g_o)\right) = 0 \\ \Rightarrow\ &D_{T(t)}\left(x^t - \left[D_{T(t)}\left(x^t, \bar{y}; (g_i, 0)\right) - H\left(y^t, g_i\right)\right].g_i, y^t; (0, g_o)\right) = 0 \end{aligned} \tag{b}$$

From Proposition 4.2:

$$\begin{aligned} &D_{T(t)}\left(x^t - \left[D_{T(t)}\left(x^t, \bar{y}; \left(g_i^t, 0\right)\right) - H\left(y^t, g_i^t\right)\right].g_i, y^t; (0, g_o)\right) \\ &= D_{T(t)}\left(\bar{x}, y^t; (0, g_o)\right) + G\left(x^t - \left[D_{T(t)}\left(x^t, \bar{y}; (g_i, 0)\right) - H\left(y^t, g_i\right)\right].g_i, g_o\right) \end{aligned} \tag{c}$$

Thus:

$$D_{T(t)}\left(\bar{x}, y^t; (0, g_o)\right) = -G\left(x^t - \left[D_{T(t)}\left(x^t, \bar{y}; (g_i, 0)\right) - H\left(y^t, g_i\right)\right].g_i, g_o\right).$$

Since the left-hand side is independent of x so is the right-hand-side. Therefore:

$$D_{T(t)}\left(\bar{x}, y^t; (0, g_o)\right) = F\left(H\left(y^t, g_i\right)\right) \tag{d}$$

where

$$F\left(H\left(y^t, g_i\right)\right) = -G\left(x^t - \left[D_{T(t)}\left(x^t, \bar{y}; (g_i, 0)\right) - H\left(y^t, g_i\right)\right].g_i, g_o\right) \tag{e}$$

is a non-increasing function. We deduce from Proposition 4.1 that:

$$D_{T(t)}\left(x^t, y^t, (g_i, 0)\right) = D_{T(t)}\left(\bar{x}, y^t, (g_i, 0)\right) - F^{-1}\left(D_{T(t)}\left(x^t, \bar{y}, (0, g_o)\right)\right). \tag{f}$$

[8] Notice that this definition differs from the definition of simultaneous input and output translation homotheticity developed in Färe and Grosskopf (2000).

Moreover:

$$\begin{aligned}
& D_{T(t)}\left(x^t, y^t, (0, g_o)\right) \\
&= \sup\left\{\delta : D_{T(t)}\left(x^t, \bar{y}, (g_i, 0)\right) - F^{-1}\left(D_{T(t)}\left(\bar{x}, y^t + \delta g_o, (0, g_o)\right)\right) \geq 0\right\} \\
&= \sup\left\{\delta : F\left(D_{T(t)}\left(x^t, \bar{y}, (g_i, 0)\right)\right) \leq D_{T(t)}\left(\bar{x}, y^t + \delta g_o, (0, g_o)\right)\right\} \\
&= \sup\left\{\delta : \delta \leq D_{T(t)}\left(\bar{x}, y^t + \delta g_o, (0, g_o)\right) - F\left(D_{T(t)}\left(x^t, \bar{y}, (g_i, 0)\right)\right)\right\} \\
&= D_{T(t)}\left(\bar{x}, y^t + \delta g_o, (0, g_o)\right) - F\left(D_{T(t)}\left(x^t, \bar{y}, (g_i, 0)\right)\right)
\end{aligned}$$

Finally, following Färe and Primont (1995) the converse is obvious. □

The second part of our proposition coincides with Proposition 2 in Fukuyama (2002). The conditions on technology in the first part imply a joint efficiency assumption. Fukuyama (2002) explains extensively the implications of this joint efficiency assumption (e.g., see his Proposition 1).

3.3 Linking Luenberger-Hicks-Moorsteen and Luenberger productivity indicators

Inspired by the work of Färe, Grosskopf and Roos (1996) linking Hicks-Moorsteen and Malmquist indices, we establish in this subsection a connection between the Luenberger-Hicks-Moorsteen and Luenberger productivity indicators.

Proposition 6. *Assume that technology $T(t)$ satisfies (T.1)–(T.4). The Luenberger-Hicks-Moorsteen productivity indicator is equal to the Luenberger output (or input) oriented productivity indicator if and only if:*

(i) the technology is inversely translation homothetic in the direction of g, and
(ii) exhibits graph translation homotheticity in the direction of g at each time period.

Proof. We define the t-based output oriented Luenberger productivity indicator by:

$$\Delta_t = D_{T(t)}\left(x^t, y^t, (0, g_o)\right) - D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) \tag{a}$$

It is easy to see that the equivalence is true if it holds for a base period t. We first show that if the t-based Luenberger-Hicks-Moorsteen indicator equals the t-based Luenberger indicator, then technology exhibits graph translation homotheticity in the direction of g. Assume that:

$$\begin{aligned}
& D_{T(t)}\left(x^t, y^t, (0, g_o)\right) - D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) \\
&= D_{T(t)}\left(x^t, y^t, (0, g_o)\right) - D_{T(t)}\left(x^t, y^{t+1}, (0, g_o)\right) \\
&\quad + D_{T(t)}\left(x^t, y^t, (g_i, 0)\right) - D_{T(t)}\left(x^{t+1}, y^t, (g_i, 0)\right)
\end{aligned} \tag{b}$$

Then:

$$\begin{aligned}
& -D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) \\
&= -D_{T(t)}\left(x^t, y^{t+1}, (0, g_o)\right) \\
&\quad + D_{T(t)}\left(x^t, y^t, (g_i, 0)\right) - D_{T(t)}\left(x^{t+1}, y^t, (g_i, 0)\right)
\end{aligned} \tag{c}$$

Next, translate by αg such that $(x^t + \alpha g_i, y^t) \in T(t)$ and $\left(x^t + \alpha g_i, y^{t+1}\right) \in T(t)$. Then:

$$\begin{aligned} & -D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) \\ = & -D_{T(t)}\left(x^t + \alpha g_i, y^{t+1}, (0, g_o)\right) \\ & + D_{T(t)}\left(x^t + \alpha g_i, y^t, (g_i, 0)\right) - D_{T(t)}\left(x^{t+1}, y^t, (g_i, 0)\right) \end{aligned} \tag{d}$$

Taking (c) and (d) yields:

$$\begin{aligned} & D_{T(t)}\left(x^t, y^{t+1}, (0, g_o)\right) - D_{T(t)}\left(x^t, y^t, (g_i, 0)\right) \\ = & D_{T(t)}\left(x^t + \alpha g_i, y^{t+1}, (0, g_o)\right) - D_{T(t)}\left(x^t + \alpha g_i, y^t, (g_i, 0)\right) \end{aligned} \tag{e}$$

Since the input-oriented directional distance function is translation homothetic:

$$D_{T(t)}\left(x^t + \alpha g_i, y^t, (g_i, 0)\right) = D_{T(t)}\left(x^t, y^t, (g_i, 0)\right) + \alpha \tag{f}$$

We deduce that

$$D_{T(t)}\left(x^t, y^{t+1}, (0, g_o)\right) + \alpha = D_{T(t)}\left(x^t + \alpha g_i, y^{t+1}, (0, g_o)\right) \tag{g}$$

And this equality implies that technology is graph translation homotheticity in the direction g. Given Corollary 1, (b) can be written:

$$\begin{aligned} & D_{T(t)}\left(x^t, y^t, (0, g_o)\right) - D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) \\ = & D_{T(t)}\left(x^t, y^t, (0, g_o)\right) - D_{T(t)}\left(x^t, y^{t+1}, (0, g_o)\right) \\ & + D_{T(t)}\left(x^t, y^t, (0, g_o)\right) - D_{T(t)}\left(x^{t+1}, y^t, (0, g_o)\right) \end{aligned} \tag{h}$$

Or, equivalently:

$$\begin{aligned} & D_{T(t)}\left(x^t, y^t, (0, g_o)\right) \\ = & D_{T(t)}\left(x^t, y^{t+1}, (0, g_o)\right) \\ & + D_{T(t)}\left(x^{t+1}, y^t, (0, g_o)\right) - D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) \end{aligned} \tag{i}$$

Since (i) holds for all $\left(x^{t+1}, y^{t+1}\right)$, we may fix this vector such that:

$$D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) = D_{T(t)}\left(\bar{x}, \bar{y}, (0, g_o)\right) = 0 \tag{j}$$

Using (j) and graph translation homotheticity, we have:

$$\begin{aligned} & D_{T(t)}\left(x^t, y^t, (0, g_o)\right) \\ = & D_{T(t)}\left(x^t, \bar{y}, (g_i, 0)\right) + D_{T(t)}\left(\bar{x}, y^t, (0, g_o)\right) \end{aligned} \tag{k}$$

implying that the technology is translation homothetic. To prove the converse, let the technology be translation homothetic and satisfy graph translation homotheticity.

In this case:

$$\begin{aligned} D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) &= D_{T(t)}\left(\bar{x}, y^{t+1}, (g_i, 0)\right) \\ &\quad + D_{T(t)}\left(x^{t+1}, \bar{y}, (0, g_o)\right) \end{aligned} \tag{l}$$

$$\begin{aligned} D_{T(t)}\left(x^{t+1}, y^{t}, (g_i, 0)\right) &= D_{T(t)}\left(x^{t+1}, y^{t}, (0, g_o)\right) \\ &= D_{T(t)}\left(\bar{x}, y^{t}, (0, g_o)\right) \\ &\quad + D_{T(t)}\left(x^{t+1}, \bar{y}, (g_i, 0)\right) \end{aligned} \tag{m}$$

$$\begin{aligned} D_{T(t)}\left(x^{t}, y^{t}, (g_i, 0)\right) &= D_{T(t)}\left(x^{t}, y^{t}, (0, g_o)\right) \\ &= D_{T(t)}\left(\bar{x}, y^{t}, (0, g_o)\right) + D_{T(t)}\left(x^{t}, \bar{y}, (g_i, 0)\right) \end{aligned} \tag{n}$$

Thus, substituting (n) in (m) and (m) in (l) we deduce:

$$\begin{aligned} & D_{T(t)}\left(x^{t}, y^{t}, (0, g_o)\right) - D_{T(t)}\left(x^{t+1}, y^{t+1}, (0, g_o)\right) \\ = & D_{T(t)}\left(x^{t}, y^{t}, (0, g_o)\right) - D_{T(t)}\left(x^{t}, y^{t+1}, (0, g_o)\right) \\ & + D_{T(t)}\left(x^{t}, y^{t}, (g_i, 0)\right) - D_{T(t)}\left(x^{t+1}, y^{t}, (g_i, 0)\right). \end{aligned}$$

Clearly,

$$\begin{aligned} & LHM_{T(t),T(t+1)}\left(x^{t}, y^{t}, x^{t+1}, y^{t+1}; g^{t}, g^{t+1}\right) \\ = & L^{i}\left(\left(x^{t}, y^{t}\right), \left(x^{t+1}, y^{t+1}\right); g_i^{t}, g_i^{t+1}\right) \\ = & L^{o}\left(\left(x^{t}, y^{t}\right), \left(x^{t+1}, y^{t+1}\right); g_o^{t}, g_o^{t+1}\right) \end{aligned}$$

□

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