Sociedad de Estadística e Investigación Operativa
Top (2006) Vol. 14, No. 1, pp. 135–166

Input, Output and Graph Technical Efficiency Measures on Non-Convex FDH Models with Various Scaling Laws: An Integrated Approach Based upon Implicit Enumeration Algorithms

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Abstract

In a recent article, Briec, Kerstens and Vanden Eeckaut (2004) develop a series of nonparametric, deterministic non-convex technologies integrating traditional returns to scale assumptions into the non-convex FDH model. They show, among other things, how the traditional technical input efficiency measure can be analytically derived for these technology specifications. In this paper, we develop a similar approach to calculate output and graph measures of technical efficiency and indicate the general advantage of such solution strategy via enumeration. Furthermore, several analytical formulas are established and some algorithms are proposed relating the three measurement orientations to one another.

Key Words: Data envelopment analysis, free disposal hull, technical efficiency.

AMS subject classification: 90B30, 90B50, 90C26.

1 Introduction

Efficiency and productivity measurement serve an important role in benchmarking firms and public sector organisations using frontier analysis. The

We are grateful to the comments made by a most constructive referee. Send all correspondence to the second author.

Manuscript received: December 2004. Final version accepted: September 2005.

boundary of technology or a corresponding economic value (e.g., cost) function can be estimated via econometric and mathematical programming methodologies (see Lovell (1993)). Non-parametric technology models, known under the name Data Envelopment Analysis (DEA), have become standard tools for efficiency measurement ever since Charnes, Cooper and Rhodes (1978) showed that efficiency measures can be computed for each Decision Making Unit (DMU) using linear programming. Starting from the first operational procedure to measure technical and allocative efficiency (Farrell (1957)), extensive efficiency and productivity decompositions have been developed (e.g., Banker, Charnes and Cooper (1984), Färe, Grosskopf and Lovell (1983)). These refined measurement schemes reveal possible causes of inferior or superior performance, which is valuable for both policy-oriented and academic purposes.

While the development of these DEA production models initially imposed convexity, Deprins, Simar and Tulkens (1984) proposed a simple monotone hull (known as the Free Disposal Hull (FDH)) as an estimator of technology. This monotone hull is the closest inner approximation of technology (Färe and Li (1998)), since it is normally contained in the convex monotone hull (e.g., Banker, Charnes and Cooper (1984)). Apart from the time divisibility of technologies, very few arguments exist to maintain convexity apart from the convenience of deriving duality results between technology and value functions (e.g., the cost function). The main argument against convexity is probably related to indivisibilities. For instance, Scarf (1981a), Scarf (1981b), Scarf (1986a), Scarf (1986b), and Scarf (1994) is among the authors stressing the importance of indivisibilities in choosing among technological options. This general argument has been used to plea in favour of using non-convex FDH technologies by Tone and Sahoo (2003). Furthermore, Briec et al. (2004) stress that our ignorance about available technological choices is especially important when analysing public sector activities, calling for a prudent formulation of technology. Furthermore, FDH has attractive statistical properties: it is a consistent estimator for any monotone boundary, though its rate of convergence is small (Simar and Wilson (2000)).

FDH has gained quite some popularity in empirical applications in a variety of sectors. Public sector studies include, among others, Drake and Simper (2003) analysing police force efficiency in the UK and Mairesse and Vanden Eeckaut (2002) assessing the performance of museums. Turning to private sector activities, Alam and Sickles (2000) estimate FDH efficiency scores to assess the dynamics of deregulation in the US airline industry and their convergence tests find less dispersion in firm performance over time, Bauer and Hancock (1993) measure the efficiency and productivity of check processing offices of the Federal Reserve System, Cummins and Zi (1998) compare the relative performance of U.S. life insurance companies, Ruiz-Torres and López (2005) use FDH to evaluate heuristics for job scheduling problems on parallel machines when there are multiple criteria, among others.

Kerstens and Vanden Eeckaut (1999) and Briec et al. (2000) proposed the integration of traditional returns to scale assumptions into this non-convex FDH model to create the closest inner approximations of technologies allowing for various scaling laws (see, e.g., Destefanis and Storti (2002) and Destefanis (2003) for empirical applications). This allows, among others, to distinguish between technical and scale efficiency on non-convex technologies too. Initially, the computation of efficiency measures relative to these models implied the solution of nonlinear, mixed integer mathematical programs. However, Podinovski (2004) showed that these problems can be linearised yielding a series of equivalent standard MIP problems. Leleu (2006) took this analysis one step further and showed how the same models can be transformed into linear programming models. But, Briec, Kerstens and Vanden Eeckaut (2004) indicate that these non-convex production frontiers need not create any computational problem in empirical applications, since simple closed-form expressions characterize the technical efficiency measures making use of implicit enumeration algorithms based upon vector comparisons. In fact, these enumeration algorithms require by far the smallest number of arithmetic operations (see also below).

In this paper, we extend this latter work and focus on output and graph oriented measures of technical efficiency. Paralleling the input case, we derive simple closed-form expressions to calculate the radial output measure of technical efficiency as well as the hyperbolic measure (see Färe et al. (1985)) that evaluates technical efficiency in the full input-output space. This is again done by making use of implicit enumeration algorithms based upon vector comparisons. Briec, Kerstens and Vanden Eeckaut (2004) also establish dual relations between these non-convex technologies allowing for various scaling laws and the corresponding cost functions. Corresponding to output and graph oriented measures of technical efficiency, one could analogously develop the non-convex revenue respectively return to the dollar profit function (see Färe et al. (2002) for the latter profit function).[1] However, due to space limitations this is beyond the scope of the current contribution.

To be more precise, the paper develops the following topics. First, traditional mathematical programming models are presented to solve for efficiency measures relative to convex technologies. Also a link is established between these traditional formulations and the generalised formulation developed in Briec et al. (2004). Second, the implicit enumeration algorithms for FDH and the other non-convex technologies are systematically treated. In addition to the input-oriented efficiency measures, we propose output- and graph-oriented efficiency measures. The latter efficiency measure simultaneously looks for reductions in inputs and expansions in outputs. This extensive discussion ends with an explicit algorithm and a simple numerical example. Third, we develop some new results regarding the relationships between these three traditional orientations of measurement. In particular, we link the optimal values of the efficiency measures and their corresponding scaling parameters under the different orientations of measurement for the non-convex technologies by developing three lemmas.

2 Technologies and Efficiency Measures

2.1 Axioms and definitions of non-convex and convex technologies[2]

Drawing upon activity analysis (Koopmans (1951)), deterministic, non-parametric technologies are based on k observations using a vector of inputs $x \in \mathbb{R}^n_+$ to produce a vector of outputs $y \in \mathbb{R}^m_+$. Technology is represented by its production possibility set $T = \{(x, y) : x \text{ can produce } y\}$, i.e., the set of all feasible input-output vectors. This work needs the following assumptions on technology:

(A.1) No free lunch (if $(x, y) \in T \wedge x = 0 \Rightarrow y = 0$) and inaction is feasible

[1] Notice that long-run profit functions are independent of convexity assumptions on technology. But, a restricted profit function (e.g., due to short-run fixed inputs or the presence of an expenditure-constraint) is not lower when tangent to a convex compared to a non-convex technology. This would imply focusing on sub-vector graph efficiency measures rather than full dimensional ones, leading to additional notational complexity.

[2] This subsection draws heavily upon Briec et al. (2004).

$((0,0) \in T)$.

(A.2) T is closed.

(A.3) Free disposal of inputs and outputs (if $(x,y) \in T \wedge (x,-y) \leq (x',-y') \Rightarrow (x',y') \in T$).

(A.4) T exhibits:

- (i) Constant Returns to Scale (CRS) ($\delta T \subseteq T$, $\forall \delta > 0$);
- (ii) Non-Increasing Returns to Scale (NIRS) ($\delta T \subseteq T$, $\forall \delta \in [0,1]$);
- (iii) Non-Decreasing Returns to Scale (NDRS) ($\delta T \subseteq T$, $\forall \delta \geq 1$);
- (iv) Variable Returns to Scale (VRS) (when (i), (ii) and (iii) do not hold).

(A.5) T is convex.

Assumptions (A.1-A.2) impose weak mathematical regularities. Free (strong) disposability of inputs (outputs) means that inputs (outputs) can be wasted without opportunity costs. Axiom (A.4) describes specific assumptions regarding the returns to scale of technologies, i.e., the scaling of production processes. The convexity assumption (A.5) is traditional, but not indispensable. Various non-parametric technologies have been derived from these axioms: e.g., the non-convex FDH (Tulkens (1993)) satisfies (A.1) to (A.3) and (A.4-iv), while the initial model of Charnes, Cooper and Rhodes (1978) satisfies (A.1) to (A.3), (A.4-i) and (A.5).

We start off from the production possibilities sets associated with a single observation and then build the technology of the sample as a union of sets. Consider a set of production units $W = \big\{(x_1,y_1),\ldots,(x_K,y_K)\big\}$ containing the null input-output vector. Individual production possibilities sets are based upon one production unit (x_k,y_k) and different maintained hypotheses of returns to scale:

$$\begin{aligned}
&S^{SD,\Gamma}(x_k,y_k) = \{(x,y) : x \geq \delta x_k, 0 \leq y \leq \delta y_k, \delta \in \Gamma\} \\
&\text{where} \qquad \Gamma \in \{VRS, CRS, NIRS, NDRS\}, \\
&\text{with} \quad \text{(i)} \quad VRS = \{\delta : \delta = 1\}; \\
&\qquad\quad \text{(ii)} \quad CRS = \{\delta : \delta \geq 0\}; \\
&\qquad\quad \text{(iii)} \quad NIRS = \{\delta : 0 \leq \delta \leq 1\}; \\
&\qquad\quad \text{(iv)} \quad NDRS = \{\delta : \delta \geq 1\}.
\end{aligned} \tag{2.1}$$

The simplest non-convex technology imposes strong disposability (A.3) and no specific scaling (i.e., variable returns to scale are imposed ($\delta = 1$)). Other technologies add a specific assumption regarding returns to scale for each single observation, whereby the scaling parameter δ follows the definitions in (A.4). Taking non-convex and convex unions of these individual production possibilities sets generates the different FDH-based technologies on the one hand and the more classic convex technologies on the other hand:

$$T^{NC,\Gamma} = \bigcup_{k=1}^{K} S^{SD,\Gamma}(x_k, y_k) \quad \text{and} \quad T^{C,\Gamma} = \text{Co}\left(\bigcup_{k=1}^{K} S^{SD,\Gamma}(x_k, y_k)\right), \quad (2.2)$$

where NC and C represent non-convexity respectively convexity, Γ is as defined in (2.1) and $\text{Co}(A)$ denotes the convex hull of a set A.

Alternatively, a unified algebraic representation of non-convex and convex technologies under different returns to scale assumptions can be written as follows:

$$\begin{aligned} T^{\Lambda,\Gamma} = \Big\{(x,y) : x \geq \sum_{k=1}^{K} x_k \delta z_k, y \leq \sum_{k=1}^{K} y_k \delta z_k,\ z_k \in \Lambda, \delta \in \Gamma\Big\}, \\ \text{where } \Lambda \in \{NC, C\}, \qquad (2.3) \\ \text{with} \quad \text{(i)} \quad NC = \Big\{z_k \in \mathbb{R}_+^K : \sum_{k=1}^{K} z_k = 1 \text{ and } z_k \in \{0,1\}\Big\}, \\ \text{(ii)} \quad C = \Big\{z_k \in \mathbb{R}_+^K : \sum_{k=1}^{K} z_k = 1 \text{ and } z_k \geq 0\Big\}, \end{aligned}$$

where Γ is again as defined before. There is one activity vector (z) operating subject to a non-convexity or convexity constraint and a scaling parameter (δ) allowing for a particular scaling of observations spanning the frontier.

2.2 Definitions of technical efficiency measures

We now introduce the definitions of the input-, output- and graph-oriented measures of technical efficiency (see Färe et al. (1985)). First, consider a radial input efficiency measure defined relative to a general non-parametric technology:

$$DF_i\left(x^\circ, y^\circ \,\middle|\, T^{\Lambda,\Gamma}\right) = \min\left\{\lambda : \lambda \geq 0, (\lambda x^\circ, y^\circ) \in T^{\Lambda,\Gamma}\right\}. \qquad (2.4)$$

It is situated between zero and unity, indicates the minimal proportional reduction of all inputs while remaining within the technology, and it has a cost interpretation.

Second, a radial output-oriented efficiency measure specifically defined to such technology is:

$$DF_o\left(x^\circ, y^\circ \left| T^{\Lambda,\Gamma}\right.\right) = \max\left\{\theta : (x^\circ, \theta y^\circ) \in T^{\Lambda,\Gamma}\right\}. \tag{2.5}$$

This radial measure is larger than unity, points out the maximal proportional expansions in all output dimensions producible from given outputs, and it has a revenue interpretation.

Third, a radial graph-oriented efficiency measure defined relative to such technology is:

$$DF_{\text{GR}}\left(x^\circ, y^\circ \left| T^{\Lambda,\Gamma}\right.\right) = \min\Big\{\lambda : \left(\lambda x^\circ, \frac{1}{\lambda} y^\circ\right) \in T^{\Lambda,\Gamma}\Big\}. \tag{2.6}$$

This hyperbolic efficiency measure indicates the minimal equiproportionate reduction in all inputs and expansion in all outputs compatible with the technology, and it has a return to the dollar profit interpretation (Färe et al. (2002)).

Efficiency computations on convex models require solving linear programming (LP) problems for each observation being evaluated (e.g., Färe et al. (1994)). Focusing for the moment on the radial input efficiency measure relative to $T^{C,\Gamma}$ requires solving for each evaluated observation (x°, y°) the following non-linear problem (P.1) based upon the convex part of the technology formulation (2.3):[3]

$$\begin{aligned} DF_i(x^\circ, y^\circ) = &\min_{\lambda,z,\delta} \lambda \\ \text{subject to} \quad &\sum_{k=1}^{K} x_{kn}\delta z_k \leq \lambda x_n^\circ, \quad n = 1, \ldots, N, \\ &\sum_{k=1}^{K} y_{km}\delta z_k \geq y_m^\circ, \quad M = 1, \ldots, M, \\ &\delta \in \Gamma, z \in C, \end{aligned} \tag{P.1}$$

[3] Analogous programming problems can be defined to solve for output and graph technical efficiency measures.

whereby Γ and NC follows the definitions in expressions (2.1) and (2.3). This mathematical programming problem can be transformed into a traditional LP as follows:

$$\begin{aligned}
DF_i(x^\circ, y^\circ) = {} & \min_{\lambda, w} \lambda \\
\text{subject to} \quad & \sum_{k=1}^{K} x_{kn} w_k \leq \lambda x_n^\circ, \quad n = 1, \ldots, N, \\
& \sum_{k=1}^{K} y_{km} w_k \geq y_m^\circ, \quad M = 1 \ldots, M, \\
& w \in \Pi, \qquad\qquad\qquad\qquad\qquad\qquad \text{(P.2)}
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad & \text{(i)} \quad \Pi \equiv \Pi^{C,CRS} = \{w_k : w_k \geq 0\}; \\
& \text{(ii)} \quad \Pi \equiv \Pi^{C,VRS} = \Big\{w_k : \sum_{k=1}^{K} w_k = 1 \text{ and } w_k \geq 0\Big\}; \\
& \text{(iii)} \quad \Pi \equiv \Pi^{C,NIRS} = \Big\{w_k : \sum_{k=1}^{K} w_k \leq 1 \text{ and } w_k \geq 0\Big\}; \text{ and} \\
& \text{(iv)} \quad \Pi \equiv \Pi^{C,NDRS} = \Big\{w_k : \sum_{k=1}^{K} w_k \geq 1 \text{ and } w_k \geq 0\Big\}.
\end{aligned}$$

The new formulation (P.1) and the traditional formulation (P.2) are linked by the following lemma.

Lemma 2.1. *Computing $DF_i(x, y)$ on convex technologies using* (P.2) *is equivalent to* (P.1).

Proof. Substitute $w_k = \delta z_k$ in (P.1), rewrite the sum constraint on the activity vector, and realise that the constraints on the scaling factor δ are in fact integrated into the latter constraint. □

Turning to non-convex technologies, radial input efficiency is computed relative to $T^{NC,\Gamma}$ by solving for each observation (x°, y°) a mixed integer, non linear programming problem (P.3):

$$\begin{aligned} DF_i(x^\circ, y^\circ) = &\min_{\lambda,z,\delta} \lambda \\ \text{subject to} \quad &\sum_{k=1}^{K} x_{kn}\delta z_k \leq \lambda x_n^\circ,\ n = 1,\ldots,N, \\ &\sum_{k=1}^{K} y_{km}\delta z_k \geq y_m^\circ,\ M = 1,\ldots,M, \\ &\delta \in \Gamma, z \in NC, \end{aligned} \tag{P.3}$$

whereby Γ and NC follows the definitions in expressions (2.1) and (2.3).

Since FDH involves no scaling, the scaling parameter (δ) is fixed at 1 in (P.3) yielding the traditional binary MIP problem. As shown in Tulkens (1993) (see also De Borger et al. (1998) and Fried et al. (1995), among others), this problem can be solved in two steps using an implicit enumeration algorithm (Garfinkel and Nemhauser (1972), § 4.5). In the first step vector dominance procedures determine for each observation its set of dominating observations (independent of the selected orientation of efficiency measurement). In the second step the efficiency measure is computed by directly applying its definition.

The algorithms developed to solve for radial efficiency measures on the new non-convex technologies are very similar in structure to the ones proposed for FDH.[4] Again, one can distinguish between two steps: the first is common to all orientations; the second is specific for each orientation of measurement. First, for each of these three orientations of measurement the set of dominating observations refers to the "scaled better set" and depends on one of the four possible returns to scale assumptions. Second, once membership of the "scaled better set" is verified, the optimal values of the scaling parameter must be substituted in the algorithms to compute the technical efficiency measure in the selected orientation.

In general, a virtue of using an enumeration approach in FDH-based models is the ability to provide algorithms requiring a relatively small number of arithmetic operations. The maximum (minimum) of a vector, whose number of components is n, can be calculated in the worst case in $O(n^2)$ arithmetic operations. Hence, enumerating on the data set (where

[4] These have also been outlined in Bogetoft (1996), page 464.

the number of firms is K) the number of arithmetic operations is about $O(LK(M+N)^2)$, where L is a measure of data storage for a given precision. By contrast, a standard linear program of a convex DEA model has a $O(LK^3)$ polynomial time complexity linked to the number of observed firms K. Since in general $K > M+N$, the time complexity of enumerative FDH models is thus better than that of DEA models. Furthermore, it is well known that binary MIP problems are computationally hard. While binary MIP models provide good empirical results in a technical efficiency analysis context due to the specific structure of the problem, their use provides certainly not the most economical way to measure firm performances. Hence, enumeration is advantageous compared to the recent proposals of Leleu (2006) and Podinovski (2004).

3 Efficiency Measures on FDH-Based Models Based upon Implicit Enumeration[5]

3.1 Scaled vector dominance

In the first step a modified index set of better observations is defined allowing for a rescaling of the observations in the sample according to the specific returns to scale assumption postulated. The vector dominance comparison thus accounts for the possibility that observations may be rescaled within certain parameter bounds. The "scaled better set" of the evaluated observation (x°, y°) is therefore conditional on one of the four returns to scale assumptions:

$$B(x^\circ, y^\circ, \Gamma) = \{(x_k, y_k) : \delta x_k \leq x^\circ, \delta y_k \geq y^\circ, \delta \in \Gamma\},$$

where Γ characterises returns to scale following (2.1). It is now obvious that we have the relationship:

$$(x_k, y_k) \in B(x^\circ, y^\circ, \Gamma) \Leftrightarrow (x^\circ, y^\circ) \in S^{SD-\Gamma}(x_k, y_k)$$

where $S^{SD-\Gamma}(x_k, y_k)$ refers to the individual production possibilities sets with different returns to scale assumptions (Γ) (i.e., expression (2.1)).

[5] To simplify notation, we assume that all observations are strictly positive (i.e., $x > 0$ and $y > 0$). To extend results to the complete non-negative Euclidean orthant, it suffices to introduce the sets: $I(x) = \{n \in \{1, \dots, N\} : x_n > 0\}$ and $J(y) = \{m \in \{1, \dots. M\} : y_m > 0\}$ (see Briec et al. (2004)).

Clearly, the construction of FDH technologies as non-convex unions of these individual subsets (expression (2.2)) makes implicit enumeration a possible solution strategy. Equivalently, this "scaled better set" can be defined by imposing lower and upper limits on the scaling parameter depending on the specific returns to scale assumption.

Thus, to obtain an enumerative process for measuring efficiency, we need to state under which conditions (x_k, y_k) "dominates" (x°, y°) given Γ.[6] To this end, the following result is needed:

Lemma 3.1. *For $k = 1, \ldots, K$, we have the following condition:*

$$(x_k, y_k) \in B(x^\circ, y^\circ, \Gamma) \Leftrightarrow \left[\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right), \min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right)\right] \cap \Gamma \neq \emptyset.$$

Proof. See Briec et al. (2004), page 168. □

Depending on the specific returns to scale assumption, this scaled dominance condition can be expressed as follows:

i) $\Gamma \equiv CRS = \{\delta : \delta \geq 0\}$: (x_k, y_k) "dominates" (x°, y°) if

$$\left[\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right), \min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right)\right] \cap [0, +\infty[\neq \emptyset.$$

Obviously, this condition reduces to

$$\left[\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right), \min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right)\right] \neq \emptyset.$$

or equivalently:

$$\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right) \leq \min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right).$$

ii) $\Gamma \equiv NIRS = \{\delta : 0 < \delta \leq 1\}$: (x_k, y_k) "dominates" (x°, y°) if

$$\left[\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right), \min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right)\right] \cap [0, 1] \neq \emptyset.$$

[6] Cherchye et al. (2001) do not need vector dominance, but instead use complete enumeration on FDH given their choice of a particular efficiency measure, known as the gauge function.

This condition implies:

$$\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right) \leq 1.$$

iii) $\Gamma \equiv NDRS = \{\delta : \delta \geq 1\}$: (x_k, y_k) "dominates" (x°, y°) if

$$\left[\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right), \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right] \cap [1, \infty[\neq \emptyset.$$

This immediately implies:

$$1 \leq \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right).$$

iv) $\Gamma \equiv VRS = \{\delta : \delta = 1\}$: (x_k, y_k) "dominates" (x°, y°) if

$$\left[\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right), \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right] \cap \{1\} \neq \emptyset.$$

This condition is rewritten:

$$\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right) \leq 1.$$

and

$$1 \leq \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right).$$

which amounts to traditional FDH dominance: $x_k \leq x^\circ$, $y_k \geq y^\circ$.

Vector dominance has been defined in terms of input and output ratios over all dimensions. Clearly, the set of dominating observations in FDH is a special case where scaling is not allowed ($\delta = 1$).

3.2 Computing efficiency measures using scaled vector dominance

In the second step efficiency measures in the input, the output or the graph orientation can be computed given some knowledge about the scaling parameter. It is not necessary to test for all possible values of the scaling parameter (δ). Instead, for each observation being evaluated one only needs to find optimal values for this scaling parameter depending on the selected orientation of measurement and the assumption made regarding returns to scale. We treat sequentially the case of the input, the output and the graph orientation of measurement. We provide a simple formula for each selected orientation.

3.2.1 Input-oriented radial efficiency measure

From the enumerative principle and the above definition (2.4) of the input-oriented radial efficiency measure, the minimum of the union set is the smallest minimum achieved over each of the separate sets. Thus, we have the following property:

$$DF_i\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \min\left\{DF_i\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right) : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

From this property, it is straightforward to state the following result:

Lemma 3.2. *(1) Under CRS or NIRS, we have:*

$$DF_i\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \min\left\{\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right) \cdot \max_{n=1,\dots,N}\left(\frac{x_{kn}}{x_n^\circ}\right) : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

(2) Under NDRS, we have:

$$DF_i\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \min\left\{\max\left(\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right), 1\right) \cdot \max_{n=1,\dots,N}\left(\frac{x_{kn}}{x_n^\circ}\right) : (x_k, y_k) \in B\left(x^\circ, y^\circ, \Gamma\right)\right\}.$$

(3) Under VRS, we have:

$$DF_i\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \min\left\{\max_{n=1,\dots,N}\left(\frac{x_{kn}}{x_n^\circ}\right) : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

Proof. See Briec et al. (2004), pages 185-186. □

3.2.2 Output-oriented radial efficiency measure[7]

As in the previous subsection, by using the enumerative principle and the above definition (2.5) of the output-oriented radial efficiency measure, the

[7] To obtain output-orientated efficiency measures contained in the unit interval, the ratios in the output dimensions in the second step of each formula must basically be reversed.

maximum over the union set is the greatest maximum achieved over each of the separate sets. Thus, we again obtain the property:

$$DF_o\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \max\left\{DF_o\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right) : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

From this property, it is straightforward to state the following result with a method of proof similar to the previous one.

Lemma 3.3. *(1) Under CRS or NDRS, we have:*

$$DF_o\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \max\left\{\min_{m=1,\ldots,M}\left(\frac{y_{km}}{y_m^\circ}\right) \cdot \min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right) : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

(2) Under NIRS, we have:

$$DF_o\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \max\left\{\min_{m=1,\ldots,M}\left(\frac{y_{km}}{y_m^\circ}\right) \cdot \min\left(1, \min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right) : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

(3) Under VRS, we have:

$$DF_o\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \max\left\{\min_{m=1,\ldots,M}\left(\frac{y_{km}}{y_m^\circ}\right) : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

Proof. See the Appendix. □

3.2.3 Graph-oriented radial efficiency measure

Again the enumerative principle and the above definition (2.6) of the graph-oriented radial efficiency measure guarantee that the minimum over the union set is the smallest minimum achieved over each of the separate sets, yielding the property:

$$DF_{GR}\left(x^\circ, y^\circ \left| T^{NC-\Gamma}\right.\right) = \min\left\{DF_{GR}\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right) : (x_k, y_k) \in \mathrm{B}(x^\circ, y^\circ, \Gamma)\right\}.$$

The following result again follows suit.

Lemma 3.4. *(1) Under CRS, we have:*

$$DF_{GR}\left(x^\circ, y^\circ \,\middle|\, T^{NC-\Gamma}\right) =$$
$$\min\left\{\left(\max_{m=1,\ldots,M}\left(\frac{y_m^\circ}{y_{km}}\right)\Big/\min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right)^{1/2} : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

(2) Under NIRS or NDRS, we have:

$$DF_{GR}\left(x^\circ, y^\circ \middle| T^{NC-\Gamma}\right) =$$
$$\min\left\{DF_{GR}^{+}\left(x^\circ, y^\circ \middle| S^{SD-\Gamma}(x_k, y_k)\right) : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}$$

where

$$DF_{GR}^{+}\left(x^\circ, y^\circ \middle| S^{SD-\Gamma}(x_k, y_k)\right) =$$

$$\begin{cases} \left(\max_{m=1,\ldots,M}\left(\frac{y_m^\circ}{y_{km}}\right)\Big/\min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right)^{1/2} & \textit{when } \alpha_k \in \Gamma \\ \max\left(\left(\min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right)^{-1}, \max_{m=1,\ldots,M}\left(\frac{y_m^\circ}{y_{km}}\right)\right) & \textit{otherwise;} \end{cases}$$

and

$$\alpha_k = \left(\min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right) \cdot \max_{m=1,\ldots,M}\left(\frac{y_m^\circ}{y_{km}}\right)\right)^{1/2}.$$

(3) Under VRS, we have:

$$DF_{GR}\left(x^\circ, y^\circ \,\middle|\, T^{NC-\Gamma}\right) = \min\left\{\max\left\{DF_i\left(x^\circ, y^\circ \middle| S^{SD-\Gamma}(x_k, y_k)\right),\right.\right.$$
$$\left.\left.\left(DF_o\left(x^\circ, y^\circ \middle| S^{SD-\Gamma}(x_k, y_k)\right)\right)^{-1}\right\} : (x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)\right\}.$$

Proof. See the Appendix. □

3.3 Conclusions

This section has shown the possibility to develop closed-form expression for the three basic orientations in efficiency measurement. Apart from the input orientation (reported in Briec et al. (2004) and repeated for the sake of completeness), these results are new. Briec et al. (2004) also develop closed-form expressions for the dual cost function, the ray-average cost function and the marginal cost function corresponding to the above input efficiency measure. In principle, corresponding to output- and graph-oriented technical efficiency measures, one could analogously develop the non-convex revenue respectively return to the dollar profit function, as well as their ray-average and marginal counterparts. We refrain from doing so for reasons of space.

Note that it is equally well possible to compute in an analogous way nonradial efficiency measures in the three orientations of measurement that allow to accommodate for the massive presence of slacks in FDH-type technologies (see De Borger et al. (1998) and Portela et al. (2003) for the traditional FDH case). Furthermore, it is in principle feasible to derive similar expressions for the directional distance function that is dual to the traditional profit function (Chambers et al. (1998)). This directional distance function can be related to the traditional Shephardian distance functions (thus, to the input- and output-oriented efficiency measures, though not to the graph-oriented technical efficiency measure). Cherchye et al. (2001) have developed a simplified enumeration algorithm for the gauge function, another specific type of efficiency measure looking for simultaneous improvements in both inputs and outputs. (see also footnote 5).

Mairesse and Vanden Eeckaut (2002) show that enumeration remains possible when defining a more restricted returns to scale (RRS) notion (hence, FDH-RRS), based upon an interval setting a lower and upper bound defining the range within which an observation can be scaled downwards or upwards. Apart from these modifications in returns to scale, enumeration remains feasible when additional constraints are added to the standard production problems treated here. For instance, one can think of the model of Färe et al. (1990) on profit maximisation subject to an expenditure constraint.

In the end, the only inconvenience one could think of is that for each type of objective function (some type of efficiency measure or value function), one must come up with a specific closed-form expression. However, notice that the first step of scaled vector dominance is only depending on the returns to scale assumption and not on any specific objective function.

4 Algorithm and Numerical Example

We are now able to outline a simple algorithm to compute the three orientations of the above efficiency measure on FDH technologies. Concentrating on the radial input efficiency measure, the following algorithm computes the input efficiency of observation (x°, y°) on FDH-based models:

[1] Choose Γ and $(x^\circ, , y^\circ)$. Let $D = 1$ (initialisation).

[2] For CRS and $NIRS$, at step k compute $\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right)$ and $\min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right)$.

- If
$$\left[\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right), \min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right)\right] \cap \Gamma = \emptyset,$$
then $(x_k, y_k) \notin B(x^\circ, y^\circ, \Gamma)$.
If $k = K$, then stop; otherwise $k := k + 1$.
- Else if
$$\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right) \cdot \max_{n=1,\ldots,N}\left(\frac{x_{kn}}{x^\circ_n}\right) < D,$$
then do:
$$D = \max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right) \cdot \max_{n=1,\ldots,N}\left(\frac{x_{kn}}{x^\circ_n}\right),$$
If $k = K$, then stop. Note: when the algorithm stops: $D = DF_i\left(x^\circ, y^\circ \middle| T^{NC-\Gamma}\right)$. Otherwise, $k := k + 1$.

For $NDRS$, at step k compute $\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right)$, $\max\left(\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right), 1\right)$ and $\min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right)$.

- If
$$\left[\max_{m=1,\ldots,M}\left(\frac{y^\circ_m}{y_{km}}\right), \min_{n=1,\ldots,N}\left(\frac{x^\circ_n}{x_{kn}}\right)\right] \cap [1, +\infty[= \emptyset,$$
then $(x_k, y_k) \notin B(x^\circ, y^\circ, \Gamma)$.
If $k = K$, then stop; otherwise $k := k + 1$.

- Else if $\max_{n=1,\dots,N}\left(\frac{x_{kn}}{x_n^\circ}\right)\cdot\max\left(\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right),1\right) < D$, then do:

$$D=\max_{n=1,\dots,N}\left(\frac{x_{kn}}{x_n^\circ}\right)\cdot\max\left(\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right),1\right)$$

If $k=K$, then stop. Note: see above. Otherwise, $k:=k+1$.

For VRS, at step k compute $\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right)$ and $\min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)$.

- If

$$\left[\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right),\min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right]\cap\{1\}=\emptyset$$

(or equivalently: $x_k \not\leq x^\circ$ or $y_k \not\geq y^\circ$), then $(x_k,y_k)\notin B(x^\circ,y^\circ,\Gamma)$. If $k=K$, then stop; otherwise $k:=k+1$.

- Else if $\max_{n=1,\dots,N}\left(\frac{x_{kn}}{x_n^\circ}\right) < D$, then do:

$$D=\max_{n=1,\dots,N}\left(\frac{x_{kn}}{x_n^\circ}\right).$$

If $k=K$, then stop. Note: see above. Otherwise, $k:=k+1$.

The computation of output- and graph-oriented efficiency measures proceeds along similar lines.

Figure 1 illustrates this algorithm using a simple numerical example with one input producing a single output. First, we clarify the role of the optimal values of the scaling parameter. Then, we indicate the computation of efficiency.

Observation a is not dominated in the traditional sense ($B(x^{\mathrm{a}},y^{\mathrm{a}},\Gamma)=\emptyset$), since no observation uses less inputs to produce more outputs (is situated in the region to the north-west of a). When CRS is assumed, then observation a is dominated by a rescaling of observations b and c (the line segments b′b″ and c′c″). When $NIRS$ ($NDRS$) are postulated, then observation a is dominated by a rescaling of observation c (b). Dividing the figure in four quadrants originating in DMU a, this observation can clearly never be dominated by rescaling any observation (like e) in the south-east region. For the other three quadrants, only observations located above the ray from the origin to DMU a are potential members of $B(x^{\mathrm{a}},y^{\mathrm{a}},\Gamma)$. For instance, observation a is not dominated by any rescaling of observation d.

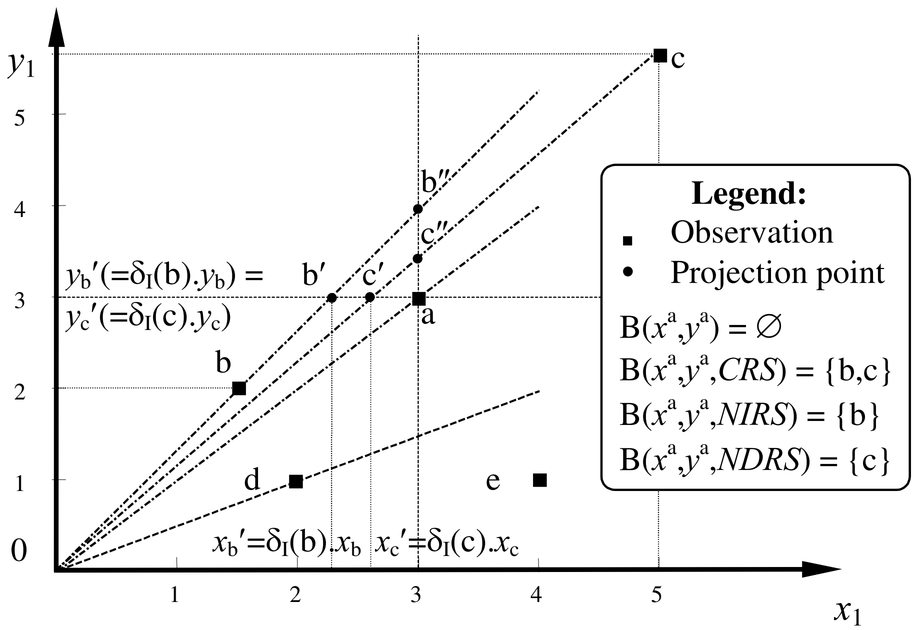

Figure 1: *Efficiency on Non-Convex Technologies: Intuition behind the Algorithm*

Observe that the input optimal value is equivalent to a projection in the output orientation. Assume observation a is evaluated imposing CRS, then the outputs of observation b (c) have to be adjusted upwards (downward) to b′ (c′) before it starts dominating DMU a. The scaling parameter for b is 3/2 and for c it is 3/5.5. The co ordinates of $\text{b}' = (x'_\text{b}, y'_\text{b})$ are $(3/2 \cdot 3/2, 3/2 \cdot 2) = (9/4, 2)$ and those of $\text{c}' = (x'_\text{c}, y'_\text{c})$ are $(3/(5.5) \cdot 5, 3/(5.5) \cdot (5.5)) = (15/(5.5), 3)$. In case $NIRS$ ($NDRS$) would be imposed the scaling parameter of b (c) would exceed (fall short of) the upper (lower) bound of 1. Therefore, b (c) would not be part of $B(x^\text{a}, y^\text{a}, NIRS)(B(x^\text{a}, y^\text{a}, NDRS))$.

Once the optimal value of the scaling parameter satisfies the bounds, the efficiency measures can be straightforwardly computed by direct application of the algorithm in expression (12) in the main text. Taking ratios of rescaled inputs for b and c over inputs of a, one directly sees that the ratios with respect to b yield the minimum ([(9/4)/3] < [(15/(5.5))/3]). Hence, DMU a is projected onto point b′ on the CRS frontier. Similar reasoning applies for the other returns to scale frontiers.

5 Relations between Different Measurement Orientations

Note that in the proof of part (3) of the previous lemma, we have stated for an individual technology that:

$$DF_{GR}\big(x^\circ, y^\circ \big| S^{SD}(x_k, y_k)\big) = \\ \max\left\{DF_i\big(x^\circ, y^\circ \big| S^{SD}(x_k, y_k)\big), \big(DF_o\big(x^\circ, y^\circ \big| S^{SD}(x_k, y_k)\big)\big)^{-1}\right\}.$$

For a general production technology, however, only an inequality holds. In particular, Färe et al. (1985), pages 136-137, were the first to prove that under a free disposal assumption the following relation holds between the three measurement orientations:

$$DF_{GR}\big(x^\circ, y^\circ \big| T\big) \geq \max\left\{DF_i\left(x^\circ, y^\circ \big| T\right), \left(DF_o\left(x^\circ, y^\circ \big| T\right)\right)^{-1}\right\}.$$

An obvious question of interest is whether any relation can be established between the optimal values of the efficiency measures and their corresponding scaling parameters under the different orientations of measurement. We answer this question by developing three more lemmas.

Lemma 5.1. *For all technologies satisfying strong disposability assumptions, we have:*

(a) $DF_i\left(x^\circ, y^\circ \left| T\right.\right) \leq DF_{GR}\left(x^\circ, y^\circ \left| T\right.\right) \leq \left(DF_i\left(x^\circ, y^\circ \left| T\right.\right)\right)^{-1}$*;*

(b) $\left(DF_o\left(x^\circ, y^\circ \left| T\right.\right)\right)^{-1} \leq DF_{GR}\left(x^\circ, y^\circ \left| T\right.\right) \leq DF_o\left(x^\circ, y^\circ \left| T\right.\right)$*.*

Proof. See the Appendix. □

Lemma 5.2. *Efficiency measures on the FDH-type of technologies can be rewritten as follows:*

(a) $DF_i\left(x^\circ, y^\circ \left| T^{NC,\Gamma}\right.\right) = \min\left\{\lambda : \lambda \geq 0, (\lambda x^\circ, y^\circ) \in \delta\, T^{NC,VRS}, \delta \in \Gamma\right\}$

(b) $DF_o\left(x^\circ, y^\circ \left| T^{NC,\Gamma}\right.\right) = \max\left\{\theta : (x^\circ, \theta y^\circ) \in \delta\, T^{NC,VRS}, \delta \in \Gamma\right\}$.

(c) $DF_{GR}\left(x^{\circ}, y^{\circ} \left| T^{NC,\Gamma}\right.\right) = \min\left\{\lambda : (\lambda x^{\circ}, \frac{1}{\lambda} y^{\circ}) \in \delta\, T^{NC,VRS}, \delta \in \Gamma\right\}$.

Proof. This is a consequence of the definition of the scaling laws (A.4). □

This result basically serves to simplify the formulation of the following lemma. For the three measurement orientations, this lemma establishes the relation between optimal values of efficiency measures on the one hand and the optimal values of the scaling parameters on the other hand.

Lemma 5.3. *Denoting:*

$$(\lambda_i^*, \delta_i^*) = \arg\min\left\{\lambda : \lambda \geq 0, (\lambda x^{\circ}, y^{\circ}) \in \delta\, T^{NC,VRS}, \delta \in \Gamma\right\};$$

$$(\theta_o^*, \delta_o^*) = \arg\max\left\{\theta : (x^{\circ}, \theta y^{\circ}) \in \delta\, T^{NC,VRS}, \delta \in \Gamma\right\};$$

$$(\lambda_{GR}^*, \delta_{GR}^*) = \arg\min\left\{\lambda : (\lambda x^{\circ}, \frac{1}{\lambda} y^{\circ}) \in \delta\, T^{NC,VRS}, \delta \in \Gamma\right\}.$$

the following relations hold true:

(a) $\lambda_i^* \leq \lambda_{GR}^* \leq \theta_{\circ}^*$;

(b) $\delta_i^* \leq \delta_{GR}^* \leq \delta_{\circ}^*$.

Proof. See the Appendix. □

While the result for the efficiency measures (part (a)) simply confirms the results valid for general strongly disposable technologies (see Lemma 5.1), the results on the optimal scaling parameters are new. They indicate that the optimal scaling needed to be dominated in terms of scaled vector dominance depends on the selected orientation of measurement and the presumed objectives pursued by the DMU's, a choice made by the modeller. This is also of some relevance for practitioners in view of recent discussions on the need to impose limitations on the scaling of DMU's in view of the supposedly unrealistic nature of the traditional scaling laws (see Mairesse and Vanden Eeckaut (2002) and the discussion supra, or Bouhnik et al. (2001), Petersen (2001), Thore (1996), amongst others).

It is useful to clarify the role of the optimal values of the scaling parameter by reference to Figure 2 that is similar in structure to Figure 1. The input optimal value is equivalent to a projection in the output orientation,

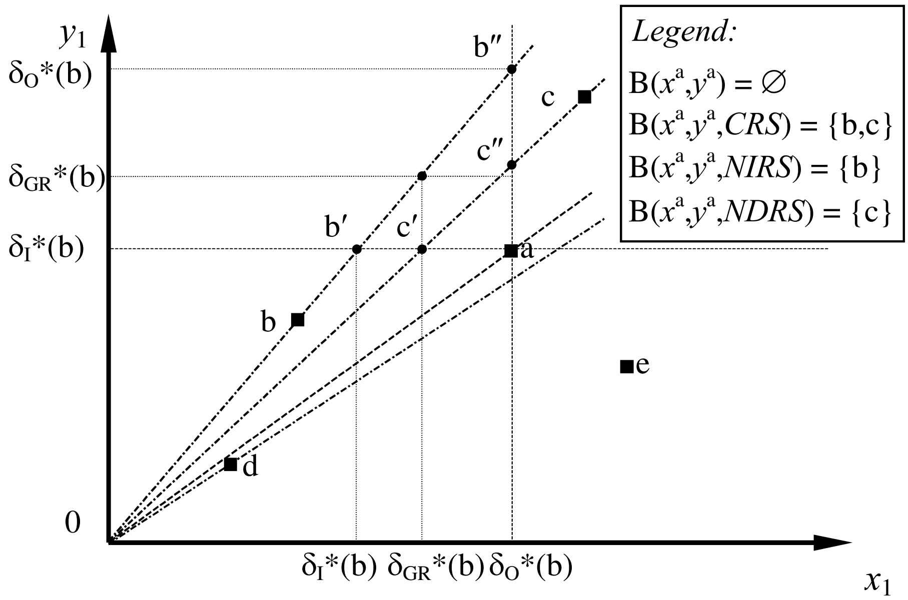

Figure 2: *The Better Set, the Scaled Better Set and Optimal Values of the Scaling Parameter*

since the outputs of observation b have to be adjusted before it starts dominating the evaluated observation a. Similarly, the output optimal value is equivalent to the projection in the input orientation (inputs need adjustment), and the graph optimal value is equivalent to the projection in the graph orientation (inputs and outputs need adjustment). Also these values are illustrated for observation b.

6 Conclusions

This contribution has extended the current literature on computing efficiency measures relative to nonparametric, non-convex technologies presented in Kerstens and Vanden Eeckaut (1999) and Briec et al. (2000). Rather than solving the original nonlinear, mixed integer mathematical programs, linearised MIP programs (as in Podinovski (2004)), or standard linear programming models (as in Leleu (2006)), we have argued that algorithms based upon implicit enumeration can in principle do the job for each efficiency measures in any type of measurement orientation. Closed-form expressions have been developed for radial input, output and graph efficiency measures and an algorithm has been designed. Furthermore, some relations between efficiency measures and scaling parameters under the three measurement orientations have been established. This should contribute to give the empirical researcher a wider choice of computing options.

Appendix

Proof of Lemma 3.3.

First, assuming that $(x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)$, compute $DF_\circ\big(x^\circ, y^\circ| S^{SD-\Gamma}(x_k, y_k)\big)$. We have $\delta x_k \leq x^\circ$, $\delta \in \Gamma$. This implies $\delta \leq \min_{n=1,\dots,N}\Big(\frac{x_n^\circ}{x_{kn}}\Big)$, $\delta \in \Gamma$. This directly yields:

$$DF_o\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right) =$$

$$\max\left\{\theta : \theta y^\circ \leq \delta y_k, \delta \leq \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right), \delta \in \Gamma\right\} =$$

$$\max\left\{\theta : \theta \leq \min_{m=1,\dots,M}\left(\frac{y_{km}}{y_m^\circ}\right)\cdot\delta, \delta \leq \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right), \delta \in \Gamma\right\}.$$

One must distinguish between two cases: (1) Under CRS or $NDRS$, the upper bound of the set $\left[\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right), \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right] \cap \Gamma$ is necessarily $\delta_{\text{sup}} = \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)$. Consequently, we obtain

$$DF_o\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right) = \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\cdot \min_{m=1,\dots,M}\left(\frac{y_{km}}{y_m^\circ}\right)$$

and from the enumerative principle, this terminates the proof. (2) Under $NIRS$, the upper bound is $\delta_{\text{sup}} = \min\Big(1, \min_{n=1,\dots,N}\left(\frac{x_m^\circ}{x_{km}}\right)\Big)$. Clearly, $\delta_{\text{sup}} = 1$ when

$$\left[\max_{m=1,\dots,M}\left(\frac{y_m^\circ}{y_{km}}\right), \min_{n=1,\dots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right] \not\subset [0, 1].$$

Consequently,

$$DF_o\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right) = \min_{m=1,\ldots,M}\left(\frac{y_{km}}{y_m^\circ}\right)\cdot\min\left(1, \min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right).$$

This completes the proof from the enumerative principle. (3) Under VRS, similar arguments lead to the result that

$$DF_o\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right) = \min_{m=1,\ldots,M}\left(\frac{y_{km}}{y_m^\circ}\right)$$

and again part (3) follows immediately.

Proof of Lemma 3.4.

To prove (1), assume that $(x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)$ and calculate

$$DF_{GR}\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right).$$

From the definition of the graph efficiency measure, one can deduce

$$DF_{GR}\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right) = \min\left\{\lambda : \delta \leq \lambda \min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right), \lambda \geq \frac{1}{\delta}\max_{m=1,\ldots,M}\left(\frac{y_m^\circ}{y_{km}}\right), \delta \in \Gamma\right\}.$$

The constraints in the mathematical programming problem are:

$$\delta \leq \lambda \min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right) \tag{A.1}$$

$$\lambda \geq \frac{1}{\delta}\max_{m=1,\ldots,M}\left(\frac{y_m^\circ}{y_{km}}\right) \tag{A.2}$$

$$\delta \geq 0. \tag{A.3}$$

Combining inequalities (A.1) and (A.2), we obtain

$$\lambda^2 \geq \max_{m=1,\ldots,M}\left(\frac{y_m^\circ}{y_{km}}\right)\bigg/\min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right).$$

It is straightforward to deduce the solutions of the optimisation program:

$$\lambda^* = \left(\max_{m=1,\ldots,M}\left(\frac{y_m^\circ}{y_{km}}\right)\bigg/\min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right)^{1/2}$$

and

$$\delta^* = \left(\min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2}.$$

This terminates part (1).

To prove (2), we first consider the $NIRS$ case with constraints:

$$\delta \leq \lambda \min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \tag{A.4}$$

$$\lambda \geq \frac{1}{\delta} \max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \tag{A.5}$$

$$0 \leq \delta \leq 1. \tag{A.6}$$

There are now two possibilities.

i) $\left(\min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2} \leq 1$. In this case the NIRS constraint does not affect the optimal solution and this again results in

$$\lambda^* = \left(\max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \Big/ \min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \right)^{1/2}$$

and

$$\delta^* = \left(\min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2}.$$

ii) $\left(\min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2} > 1$. Since $0 \leq \delta \leq 1$, one can show that at the optimum $\delta^* = 1$ (because if (A.4) and (A.5) are binding it is easy to verify that $\delta^* = \left(\min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2} > 1$ and this would not satisfy the $NIRS$ constraint). Consequently, $1 \leq \lambda \min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right)$ and $\max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \leq \lambda$. This implies

$$\lambda^* = \max \left(\left(\min_{n=1,\dots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \right)^{-1}, \max_{m=1,\dots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right),$$

and (2) follows.

Then, we treat the $NDRS$ case with constraints:

$$\delta \leq \lambda \min_{n=1,\ldots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \tag{A.7}$$

$$\lambda \geq \frac{1}{\delta} \max_{m=1,\ldots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \tag{A.8}$$

$$\delta \geq 1 \tag{A.9}$$

Again we have to consider two cases:

i) $\left(\min_{n=1,\ldots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\ldots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2} \geq 1$. In this case the $NDRS$ constraint does not affect the optimal solution and we obtain again

$$\lambda^* = \left(\max_{m=1,\ldots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \Big/ \min_{n=1,\ldots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \right)^{1/2}$$

and

$$\delta^* = \left(\min_{n=1,\ldots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\ldots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2}.$$

ii) $\left(\min_{n=1,\ldots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\ldots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2} < 1$. Since $\delta \geq 1$, the optimum $\delta^* = 1$ results (because if (A.7) and (A.8) are binding,

$$\delta^* = \left(\min_{n=1,\ldots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \cdot \max_{m=1,\ldots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right)^{1/2} < 1$$

and it would not satisfy the $NDRS$ constraint). Consequently,

$$1 \leq \lambda \min_{n=1,\ldots,N} \left(\frac{x_n^\circ}{x_{kn}} \right)$$

and

$$\max_{m=1,\ldots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \leq \lambda.$$

This implies

$$\lambda^* = \max \left(\left(\min_{n=1,\ldots,N} \left(\frac{x_n^\circ}{x_{kn}} \right) \right)^{-1}, \max_{m=1,\ldots,M} \left(\frac{y_m^\circ}{y_{km}} \right) \right).$$

By synthesising the $NIRS$ and $NDRS$ results, we then obtain (2).

Finally, to prove (3), assume that $(x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)$ and compute $DF_{GR}\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right)$. The constraints in the VRS case read:

$$\lambda x_n^\circ \geq x_{kn}$$

$$\frac{1}{\lambda} y_m^\circ \leq y_{km}.$$

These can be rewritten as:

$$\lambda \geq \left(\min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right)^{-1}$$

$$\lambda \geq \left(\min_{m=1,\ldots,M}\left(\frac{y_{km}}{y_m^\circ}\right)\right)^{-1}.$$

Since

$$\left(\min_{n=1,\ldots,N}\left(\frac{x_n^\circ}{x_{kn}}\right)\right)^{-1} = \max_{\mathrm{n=1,\ldots,N}}\left(\frac{x_{\mathrm{kn}}}{x_{\mathrm{n}}^\circ}\right) = DF_i\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right)$$

and

$$\left(DF_o\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right)\right)^{-1} = \left(\min_{m=1,\ldots,M}\left(\frac{y_{km}}{y_m^\circ}\right)\right)^{-1},$$

we immediately deduce that

$$\lambda^* = \max\left\{DF_i\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right), \left(DF_o\left(x^\circ, y^\circ \left| S^{SD-\Gamma}(x_k, y_k)\right.\right)\right)^{-1}\right\}$$

and part (3) is proven.

Proof of Lemma 5.1.

(a) We show the second inequality. For any $\lambda \leq 1$, by free disposal, it is clear that $\left(\lambda x^\circ, \frac{1}{\lambda} y^\circ\right) \in T$ implies that $\left(x^\circ, \frac{1}{\lambda} y^\circ\right) \in T$ (because $\lambda x^\circ \leq x^\circ$). From the definition of output and graph oriented measures, it is now obvious to deduce that $DF_i\left(x^\circ, y^\circ \left| T\right.\right) \leq DF_{GR}\left(x^\circ, y^\circ \left| T\right.\right)$. The second inequality is obvious because $\left(DF_i\left(x^\circ, y^\circ \left| T\right.\right)\right)^{-1} > 1$. (b) Because of the free disposal assumptions, at the optimum $\frac{1}{DF_{GR}(x^\circ, y^\circ |T)} y^\circ \leq DF_o\left(x^\circ, y^\circ \left| T\right.\right) y^\circ$. Thus, $\left(DF_o\left(x^\circ, y^\circ \left| T\right.\right)\right)^{-1} \leq DF_{GR}\left(x^\circ, y^\circ \left| T\right.\right)$. Moreover, we obviously have $DF_o\left(x^\circ, y^\circ \left| T\right.\right) \geq 1$ and $DF_{GR}\left(x^\circ, y^\circ \left| T\right.\right) \leq 1$. Consequently, we obtain $DF_{GR}\left(x^\circ, y^\circ \left| T\right.\right) \leq DF_o\left(x^\circ, y^\circ \left| T\right.\right)$ and (b) is proven.

Proof of Lemma 5.3.

(a) follows from lemma 6. Let us show (b). First, assume that $(x_k, y_k) \in B(x^\circ, y^\circ, \Gamma)$ and let us show the result for each individual technology $S^{SD-\Gamma}(x_k, y_k)$. Note that $(\lambda_i^* x^\circ, y^\circ) \in \delta_i^* S^{SD}(x_k, y_k)$, where $S^{SD}(x_k, y_k)$ is an individual strongly disposable production possibilities set. This condition obviously means that

$$DF_i\left(x^\circ, y^\circ \left| S^{SD}(x_k, y_k)\right.\right) = \min\left\{\lambda : \lambda \geq 0, (\lambda x^\circ, y^\circ) \in \delta_i^*\, S^{SD}(x_k, y_k)\right\} = \lambda_i^*.$$

Now let us denote:

$$\begin{aligned} D_i\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) &= \min\left\{\lambda : \lambda \geq 0, (\lambda x^\circ, y^\circ) \in \delta_i^*\, S^{SD}(x_k, y_k)\right\}; \\ D_o\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) &= \max\left\{\theta : (x^\circ, \theta y^\circ) \in \delta_i^*\, S^{SD}(x_k, y_k)\right\}; \\ D_{GR}\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) &= \min\left\{\lambda : (\lambda x^\circ, \tfrac{1}{\lambda} y^\circ) \in \delta_i^*\, S^{SD}(x_k, y_k)\right\}. \end{aligned}$$

Note in particular that δ_i^* is present in each of these expressions. Obviously, $D_i\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) = DF_i\left(x^\circ, y^\circ \left| S^{SD}(x_k, y_k)\right.\right) = \lambda_i^*$. From the structure of the individual observation technology $S^{SD}(x_k, y_k)$, we have:

$$D_{GR}\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) = \max\left\{D_i\left(x^\circ, y^\circ \left|\delta_i^*\right.\right), \left(D_o\left(x^\circ, y^\circ \left|\delta_i^*\right.\right)\right)^{-1}\right\}.$$

To prove that $\delta_i^* \leq \delta_{GR}^*$, we need to show that there does not exist some $\tilde{\delta} \in \Gamma$ with $\tilde{\delta} < \delta_i^*$, such that $D_{GR}\left(x^\circ, y^\circ \left|\tilde{\delta}\right.\right) < D_{GR}\left(x^\circ, y^\circ \left|\delta_i^*\right.\right)$. We have to consider two cases:

(i) $D_{GR}\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) = D_i\left(x^\circ, y^\circ \left|\delta_i^*\right.\right)$. In such a case, the input constraints are binding. Assume that there exists some $\tilde{\delta} \in \Gamma$ such that $\tilde{\delta} < \delta_i^*$ and $D_{GR}\left(x^\circ, y^\circ \left|\tilde{\delta}\right.\right) = D_i\left(x^\circ, y^\circ \left|\tilde{\delta}\right.\right) < D_{GR}\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) = D_i\left(x^\circ, y^\circ \left|\delta_i^*\right.\right)$. In such a case, we immediately have $D_i\left(x^\circ, y^\circ \left|\tilde{\delta}\right.\right) < D_i\left(x^\circ, y^\circ \left|\delta_i^*\right.\right)$. But this would contradict $D_i\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) = DF_i\left(x^\circ, y^\circ \left| S^{SD}(x_k, y_k)\right.\right) = \lambda_i^*$. That is, δ_i^* would not be optimal for the input efficiency measure, a contradiction. Consequently, there does not exist some $\tilde{\delta} \in \Gamma$ with $\tilde{\delta} < \delta_i^*$, such that $D_{GR}\left(x^\circ, y^\circ \left|\tilde{\delta}\right.\right) < D_{GR}\left(x^\circ, y^\circ \left|\delta_i^*\right.\right)$.

(ii) $D_{GR}\left(x^\circ, y^\circ \left|\delta_i^*\right.\right) = \left(D_o\left(x^\circ, y^\circ \left|\delta_i^*\right.\right)\right)^{-1}$. Now for any $\tilde{\delta} \in \Gamma$ such that $\tilde{\delta} < \delta_i^*$, we have $\left\{v \in \mathbb{R}^m : v \leq \tilde{\delta} y_{km}\right\} \subset \{v \in \mathbb{R}^m : v \leq \delta_i^* y_{km}\}$. This implies that

$$D_{GR}\left(x^{\circ}, y^{\circ}\left|\tilde{\delta}\right.\right) =$$
$$\left(D_o\left(x^{\circ}, y^{\circ}\left|\tilde{\delta}\right.\right)\right)^{-1} > D_{GR}\left(x^{\circ}, y^{\circ}\left|\delta_i^*\right.\right) = \left(D_o\left(x^{\circ}, y^{\circ}\left|\delta_i^*\right.\right)\right)^{-1}.$$

Thus, there does not exist some $\tilde{\delta} \in \Gamma$ with $\tilde{\delta} < \delta_i^*$, such that $D_{GR}\left(x^{\circ}, y^{\circ}\left|\tilde{\delta}\right.\right) < D_{GR}\left(x^{\circ}, y^{\circ}\left|\delta_i^*\right.\right)$.

Since (i) and (ii) hold, there does not exist $\tilde{\delta} < \delta_i^*$, such that

$$D_{GR}\left(x^{\circ}, y^{\circ}\left|\tilde{\delta}\right.\right) > D_{GR}\left(x^{\circ}, y^{\circ}\left|\delta_i^*\right.\right).$$

Since $T^{NC-\Gamma} = \bigcup_{k=1}^{K} S^{SD-\Gamma}(x_k, y_k)$, we extend this result to the union of all individually scaled technologies. Consequently, we can deduce that $\delta_i^* \leq \delta_{GR}^*$. By using similar arguments, we can also infer that $\delta_{GR}^* \leq \delta_{\circ}^*$ and the lemma is proven.

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