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Multi-horizon Markowitz portfolio performance appraisals: A general approach☆

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Abstract

This article extends the analysis of multi-horizon mean-variance portfolio analysis in the Morey and Morey [Mutual fund performance appraisals: a multi-horizon perspective with endogenous benchmarking. Omega 1999;27:241–58] article in several ways. First, instead of either proportionally contracting risk dimensions or proportionally expanding return dimensions, a more general efficiency measure simultaneously attempts to reduce risk and to expand return over all time periods. Second, a duality relation is established between this generalized multi-horizon efficiency measure and an indirect mean-variance utility function, underscoring the natural interpretation of this generalized efficiency measure in terms of investor's preferences. Furthermore, the need to properly apply time discounting in multi-horizon mean-variance portfolio problems is argued for. An empirical illustration based on the original mutual fund data set in Morey and Morey [Mutual fund performance appraisals: a multi-horizon perspective with endogenous benchmarking. Omega 1999;27:241–58] is added to contrast the new and the original approaches.

Keywords: Portfolio selection; Mathematical programming

1. Introduction

For decades, portfolio performance has been evaluated using performance measures combining information on both return and risk, such as the Sharpe measure [1], the Treynor measure [2], the Jensen measure [3], among others (see [4] for a critical discussion).

More recently, explicit efficiency measures have been introduced to benchmark mutual funds using non-parametric frontier models (known as Data Envelopment Analysis).[1] Indeed, while these benchmarking models have been widely used to assess the relative performance of production activities in agriculture, industry and service sectors, efficiency measures based on frontier estimation are now being employed in various new domains, like marketing (e.g., [5]) and inequality measurement (e.g., [6]), to name just a few. Also, their

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[1] Kleine [12] offers a taxonomy and overview of these models, and also provides a link to the multi-criteria decision making literature.

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use in finance in general and portfolio analysis in particular is of rather recent date.[2] The seminal article employing efficiency measures to benchmark mutual funds was Murthi et al. [7], immediately followed by McMullen and Strong [8] and Premachandra et al. [9], and several others in the meantime. Within the static mean-variance portfolio selection problem, Sengupta [10] is probably the first to introduce an efficiency measure as a benchmarking tool in the quadratic optimization programs. The seminal article of Morey and Morey [11] introduces the same idea in a multi-horizon or temporal setting. This extension is important because efficiency measures, being related to distance functions, have the great advantage of being capable to position observations with respect to the boundary of multi-dimensional choice sets. Morey and Morey [11] propose two types of efficiency measures. A first efficiency measure attempts to contract all risk dimensions proportionally; a second one focuses on augmenting all return dimensions as much as possible in a proportional way.

The purpose of the current contribution is to extend the Morey and Morey [11] proposal. Multi-horizon or temporal benchmarking can in principle be applied both retrospectively and prospectively. The Morey and Morey [11] article mainly focuses on retrospective benchmarking and explicitly aims to provide an alternative weighting scheme to assess the performance over multiple periods using objective rather than subjective weights, the latter being used by some commercial rating services (in particular, Morningstar). This article follows Morey and Morey [11] in their retrospective viewpoint. But, it also briefly outlines how a prospective point of view could be developed.[3] Furthermore, the Morey and Morey [11] article takes a long-term investment perspective by benchmarking portfolios over a given time horizon. In contrast, the optimization of a periodically rebalanced benchmark over the given time horizon allows for a complementary short-term perspective. In brief, the aim of Morey and Morey [11] is first and foremost to retrospectively assess performance adapting a long-term investment perspective. The objective of our contribution is to offer in addition a short-run investment perspective. These developments require that the benchmarking framework is firmly embedded in portfolio theory.

In particular, this contribution aims to achieve the following goals. First, it shows that these efficiency measures introduced by Morey and Morey [11] are just special cases of a more general approach that simultaneously attempts to reduce risk and to expand return. Second, by stressing the importance of a dual relation between the efficiency measure (or, more generally speaking, the distance function) and an indirect mean-variance utility function, it becomes clear that the Morey and Morey [11] analysis can be refined in three important ways. First, instead of reducing all risk dimensions (or expanding all return dimensions) over a given time horizon by a common scalar, one should simply define an efficiency measure for each period within the time horizon. Second, the duality relation shows that in general one needs time discounting in multi-horizon settings, also for the efficiency measures. Indeed, given standard assumptions about time discounting, efficiency gains in the far future should be weighted less than efficiency gains in the near future when planning ahead (similar to the case of utility gains). In retrospective benchmarking, the distant past is analogously weighted less than efficiency gains in the nearby past. Third, this same duality relation also shows that this more general approach to portfolio efficiency (PE) is compatible with standard investor preferences with respect to mean-variance portfolios, while the Morey and Morey [11] efficiency measures imply extreme assumptions with respect to the risk characteristics of investors that are rather unlikely to hold in general.

Indeed, Briec et al. [17] study existing non-parametric efficiency measurement approaches for a single period mean-variance portfolio selection from a theoretical perspective and generalize the efficiency measures of Morey and Morey [11] into the full mean-variance space. Starting from a given portfolio, the generalized efficiency measure seeks for simultaneous reductions in risk and expansions in return. This generalized efficiency measure or shortage function is studied in the context of the Markowitz [31] efficient frontier, and a link is established to the indirect mean-variance utility function. This framework allows distinguishing between PE and allocative efficiency (AE). Furthermore, it permits retrieving information about the revealed

[2] Applications benchmarking production in the financial sector are widespread and include, among others, studies on the efficiency of bank branches (e.g., [13]), banks (see, e.g., the overview of [14]) and insurance companies (see [15] for a survey).

[3] Notice that retrospective and prospective benchmarking pose different informational requirements. Retrospective benchmarking is based on observed past behavior. In contrast, prospective benchmarking attempts to take account, at least partially, of likely future behavior. The required information on expected returns that can be obtained from scenario analysis or bought from specialized firms (e.g., the I/B/E/S databases of Thomson Financial based on consensus estimates). However, uncertainty surrounding the data is well-known to have an impact on portfolio optimization. For instance, estimation errors in means are more important than errors in variance–covariance matrices, whereby errors in variances weight heavier than errors in covariances (see, e.g., [16]).

risk aversion of investors. In this contribution, we basically combine this static mean-variance framework developed with the idea of a time-discounted shortage function defined in Briec et al. [18] in a temporal profit function setting. This allows evaluating the performance of a mean-variance portfolio problem over a multi-period time horizon.

This article is structured as follows. Section 2 lays the foundation by succinctly repeating the static mean-variance portfolio using the analytical tools proposed in Briec et al. [17]. A numerical example illustrates the difference between the Morey and Morey [11] approach and the generalized efficiency measure. To develop the theoretical framework, Section 3 first defines a time-discounted shortage function and a time-discounted indirect utility function in a multi-period mean-variance setting. Then, we establish a duality result between these time-discounted temporal shortage and temporal indirect mean-variance utility functions. This basically transposes the duality result in Briec et al. [18] from a production setting to a financial context. Again, a numerical example serves to illustrate the key differences between the different approaches. In the empirical application in Section 4, we exploit the original Morey and Morey [11] data. These authors use 26 mutual funds observed over the period from 1985 to 1995, and they analyze their yields in monthly averages over three time horizons. The empirical results computed and discussed are: (i) the shortage function results compared to their mean return expansion (MRE) and risk contraction (RC) efficiency measures; (ii) the discounted shortage function results compared to the standard shortage function results; and (iii) the results of the discounted overall efficiency decomposition. A concluding section outlines conclusions and eventual future extensions.

2. The static mean variance model

2.1. Theory

Developing some basic definitions, consider the problem of selecting a portfolio (or fund of funds) from n financial assets (or funds) at time period t. In each period t, assets are characterized by an expected return $E[R_i^t]$ for $i \in \{1,\dots,n\}$, and by a variance–covariance matrix $\Omega_{i,j}^t = \mathrm{Cov}[R_i^t, R_j^t]$ for $i, j \in \{1,\dots,n\}$. A portfolio $x^t = (x_1^t,\dots,x_n^t)$ is composed by a proportion of each of these n financial assets ($\sum_{i=1,\dots,n} x_i^t = 1$).

When short sales are excluded, then the condition $x_i \geqslant 0$ is imposed. Short selling is sometimes relevant for mutual funds, and even more so for hedge funds. Therefore, it is good to realize that the developed models can be extended to allow for short selling. Furthermore, when investors face additional constraints (e.g., transaction costs or upper limits on any fraction invested) that can be written as constraints that are linear functions of asset weights, then the set of admissible portfolios can be easily adapted (see [17]). In the remainder of the contribution, the basic models presented ignore short selling and any additional constraints. Of course, we do not deny that the benefits of rebalancing in our short-term investment perspective cannot be partly swamped by the existence of transaction costs, but these costs are immaterial in developing the basic benchmarking model of a periodically rebalanced fund of funds.

The return of portfolio x at the time period t is given by $R^t(x^t) = \sum_{i=1,\dots,n} x_i^t R_i^t$. The expected return and variance of this portfolio are as follows:

$$E[R^t(x^t)] = \mu^t(x^t) = \sum_{i=1,\dots,n} x_i^t E[R_i^t], \tag{2.1}$$

$$\begin{aligned}\mathrm{Var}[R^t(x^t)] &= E[(R^t(x^t) - \mu^t(x^t))^2] \\ &= \sum_{i,j} x_i^t x_j^t \,\mathrm{Cov}[R_i^t, R_j^t].\end{aligned} \tag{2.2}$$

It is useful to define the mean-variance (portfolio) *disposal representation set* through:

$$\begin{aligned}\mathrm{DR}^t = \{(V,E) \in \mathbb{R}_+^2 : \exists x^t \text{ with } V \geqslant \mathrm{Var}[R^t(x^t)],\\ E \leqslant E[R^t(x^t)]\}.\end{aligned} \tag{2.3}$$

The purpose of this set is to extend the choice of portfolio weights by allowing for some type of free disposal in all return and risk dimensions. Thus, for a given portfolio it is always possible to increase its risk and to reduce its return. This is a technical assumption facilitating the use of standard optimization tools.

In production theory, the shortage function measures —intuitively stated—the distance between some point of the production possibility set and the Pareto frontier (Luenberger [32]).[4] The following introduces the shortage function as a performance indicator for the mean-variance portfolio optimization problem (see [17]). Let x^t denote a portfolio observed at the time period t. Let $g^t = (-g_V^t, g_E^t) \in (-\mathbb{R}_+) \times \mathbb{R}_+$ be a direction vector. The shortage function $S_{g^t}^t$ measures the efficiency of each portfolio in a direction given by the vector g^t. At time period t, this function is defined for

[4] This shortage function is also known under the name directional distance function.

all portfolios x^t as:

$$S^t_{g^t}(x^t) = \sup\{\delta;\ (\mathrm{Var}[R^t(x^t)] - \delta g^t_V,\ E[R^t(x^t)] + \delta g^t_E) \in \mathrm{DR}^t\}. \tag{2.4}$$

The basic properties of the subset DR^t on which the shortage function is defined are discussed in Briec et al. [17] in the setting of Markowitz mean-variance portfolio theory. It is sufficient to stress that when $S^t_{g^t}(x^t)=0$, then the evaluated portfolio is efficient and part of the Markowitz frontier. When $S^t_{g^t}(x^t)>0$, then the portfolio is inefficient and the shortage function indicates the percentage change in terms of both return expansion and risk reduction that are needed to catch up with the portfolio frontier. For example, when $S^t_{g^t}(x^t) = 0.05$, then by augmenting return and contracting risk by 5% it is possible to join the portfolio frontier.

Briec et al. [17] demonstrate that this static shortage function generalizes both the mean return augmentation and RC efficiency measures proposed in Morey and Morey [11]. If $g^t_V = -\mathrm{Var}[R^t(x^t)]$ and $g^t_E = 0$, then setting $D_V(x^t) \equiv 1 - S_{g^t}(x^t)$ we retrieve their RC measure. Furthermore, if $g^t_V = 0$ and $g^t_E = E[R^t(x^t)]$, then $D_E(x^t) \equiv 1 + S_{g^t}(x^t)$ yields their mean MRE measure. These transformations are necessary to make the original Morey and Morey [11] efficiency measures, where 1 indicates efficiency, comparable to the shortage function, where 0 stands for efficiency.[5]

In addition to tracing the frontier of efficient portfolios, Markowitz [19] also defines an optimization program based on a mean-variance indirect utility function to determine the optimal portfolio corresponding to a given degree of risk aversion. This portfolio maximizes a dated mean-variance utility function defined by:

$$U^t_{(\rho_t,\mu_t)}(x^t) = \mu_t E[R^t(x^t)] - \rho_t \mathrm{Var}[R^t(x^t)]. \tag{2.5}$$

This utility function in time period t satisfies positive marginal utility of expected return, and negative marginal utility of risk. Briec et al. [17] prove a duality result between the above shortage function and this mean-variance indirect utility function in a static context. This confirms that the shortage function has an economic interpretation, since it is compatible with general preferences of investors over the mean-variance space.

To determine an optimal portfolio corresponding to a given degree of risk aversion within the mean-variance approach, Markowitz [19] defines a quadratic optimization program maximizing the above mean-variance utility function:

$$V(\rho_t,\mu_t) = \max \quad \mu_t E[R^t(x^t)] - \rho_t \mathrm{Var}[R^t(x^t)] \quad \text{s.t.} \sum_{i=1,\dots,n} x^t_i = 1,\ x^t \geqslant 0, \tag{2.6}$$

where the ratio $\varphi_t = \rho_t/\mu_t \in [0,+\infty]$ represents the degree of absolute risk aversion. This parameter is specific per investor, but it normally remains constant over time.

Repeating the integration of on the one hand the efficiency measure approach and on the other hand the direct utility function computation, we follow Briec et al. [17] who are the first to define an overall efficiency decomposition in the portfolio context.

Definition 2.1. Static overall efficiency decomposition:

(1) Overall efficiency (OE) index is the quantity:

$$\mathrm{OE}(x^t) = V(\rho_t,\mu_t) - U_{(\rho_t,\mu_t)}(x^t),$$

(2) Allocative efficiency (AE) index is the quantity:

$$\mathrm{AE}(x^t) = \mathrm{OE}(x^t) - S_{g^t}(x^t),$$

(3) Portfolio efficiency (PE) index is the quantity:

$$\mathrm{PE}(x^t) = S_{g^t}(x^t).$$

PE checks whether or not a portfolio is situated at the boundary of the mean-variance portfolio frontier. AE measures the gap between the portfolio projection onto the boundary and the point on the same boundary maximizing the indirect utility function. The OE notion ensures that both of these ideals are satisfied simultaneously and measures any degree of underperformance in this respect. This amounts to simply transposing the traditional distinction between technical and AE from a production to a portfolio context.

We finish this static version of the mean-variance model using the shortage function with two remarks. First, the duality between shortage function and mean-variance indirect utility allows retrieving information about the investors absolute risk aversion via the shadow prices associated with this specific efficiency measure (see Proposition 5.1 in [17]). In particular, the adjusted risk aversion function is defined as:

$$(\rho^*_t,\mu^*_t)(x^t) = \arg\min_{\rho_t,\mu_t \geqslant 0}\{V(\rho_t,\mu_t) - U_{(\rho_t,\mu_t)}(x^t) : \rho_t g^t_V + \mu_t g^t_E = 1\}. \tag{2.7}$$

[5] These transformations are analogous to the ones linking the shortage function to the traditional Shephardian distance functions, that are themselves inversely related to the radial efficiency measures employed by Morey and Morey [11]: see Chambers et al. [20].

This type of shadow indirect mean-variance utility function searches for parameters (ρ_t^*, μ_t^*) defining a shadow risk aversion that renders the current portfolio optimal for the investor. These risk aversion parameters implicitly characterize the agent's behavior by putting it in the most favorable light, i.e., by minimizing portfolio inefficiency. This (shadow) risk aversion could be employed to assess whether portfolio management strategies of, e.g., mutual funds adhere to a priori specified risk profiles.

Second, when shorting is allowed or there is a riskless asset, then it is well-known from the two- and one-fund theorems that the efficient frontier follows simple analytical solutions. However, while the computational burden of this more general quadratic programming approach remains substantial, it is unavoidable when building realistic portfolio models including cardinal restrictions on the number of assets, transaction costs, etc. (see, e.g., [21]).

Having briefly sketched the above theoretical framework, it is useful to illustrate the essential differences between our own approach and Morey and Morey [11] using a simple numerical example.

2.2. Numerical example

To construct our numerical example, we take the data from the Benninga [22, pp. 129–130] book: this author offers a simple numerical example with four assets. Mean returns and the variance–covariance matrix for these four fictitious assets are listed in Table 1.

The optimal solutions to the mean MRE approach of Morey and Morey [11] is reported in Table 2. We report on the mean MRE efficiency measure, the resulting portfolio variance and return at the frontier, and the portfolio weights. Table 3 reports the same optimal results for the shortage function approach. A glance at both tables reveals that the results are altogether rather different. Indeed, there are differences in the efficiency measure, in the frontier projections, and in the optimal portfolio weights.

It is useful to offer a more careful interpretation of these efficiency measures for one of these four fictitious assets. Asset 2 has a return of 0.09 and a variance of 0.20. Its mean MRE efficiency measure of 0.11 implies that it can increase its return by 11%. This leads to a frontier point with return 0.10 (=0.09 + 0.11 · 0.09), while its variance remains constant. The shortage function indicates an inefficiency of 0.101: thus, it could increase its return and reduce its variance by 10.1%. This results in a point of the Markowitz frontier with a return equal to 0.099 and a variance equal to 0.180. Notice that the mean MRE efficiency measures reported in Table 2 have been modified along the lines sketched above to make them comparable to the shortage function.

The differences between the different approaches can probably best be visualized in Fig. 1 which represents the original assets, the Markowitz portfolio frontier (as computed by Benninga [22]), and two frontier projections: on the one hand the mean MRE efficiency approach, and on the other hand the shortage function approach. Starting from the four basic portfolios, it is clear that the mean MRE efficiency approach leads to vertical projections onto the Markowitz frontier, while the shortage function approach projects the initial basic portfolios somewhat to the North-West onto the same Markowitz frontier. Using the direct utility function allows picking one point among the portfolio frontier, assuming that one can determine reasonable risk parameters representing the investor's preferences.

3. A temporal mean-variance model

3.1. Theory

To construct a multi-horizon or temporal mean-variance model, one first defines the notion of a temporal path for a portfolio, which is defined as $\mathcal{X} = (x^t)_{t=1}^T = \overset{T}{\underset{t=1}{\times}} x^t$. Since the direction vector may be time-dependent, we similarly assume that: $\mathcal{G} = (g^t)_{t=1}^T = \overset{T}{\underset{t=1}{\times}} g^t$. In addition, we denote $\Delta = (\delta^1, \dots, \delta^T)$ and use the convention $\Delta\mathcal{G} = (\delta_1 g^1, \dots, \delta_T g^T)$. In a discrete time framework, for a given time horizon T the mean-variance space has the dimension $\mathbb{R}_+^{2\times T}$ and consists of all sequences of dated risks and returns of the form:

$$\mathcal{F}(\mathcal{X}) = \overset{T}{\underset{t=1}{\times}} (\mathrm{Var}[R^t(x^t)], E[R^t(x^t)]). \tag{3.1}$$

We define a temporal representation set as $\mathcal{DR} = \times_{t=1}^T \mathrm{DR}^t$, that is the cartesian product of the disposal representation set at each time period mentioned previously (see Eq. (2.3)).

The definition of a temporal shortage function does the same as a static shortage function, but over multiple time periods: it simultaneously seeks to expand multiple return dimensions and contract multiple risk dimensions over all time periods. Morey and Morey [11] implicitly assume that the time dimension is neutral. But, for an economic agent the present is more valuable than the future when planning actions ahead. For retrospective benchmarking, the past is less valuable than the

Table 1
Numerical example [22]

	Means (%)	Variance–covariance matrix			
		Asset 1	Asset 2	Asset 3	Asset 4
Asset 1	8	0.10	0.03	−0.08	0.05
Asset 2	9	0.03	0.20	0.02	0.03
Asset 3	10	−0.08	0.02	0.30	0.20
Asset 4	11	0.05	0.03	0.20	0.90

Table 2
Optimal solution mean return augmentation approach

	MRE	Optimal portfolio		$x1$	$x2$	$x3$	$x4$
		Variance	Return				
Asset 1	0.181	0.100	0.094	0.207	0.256	0.421	0.116
Asset 2	0.110	0.200	0.100	0.000	0.262	0.486	0.253
Asset 3	0.029	0.300	0.103	0.000	0.078	0.555	0.368
Asset 4	0.000	0.900	0.110	0.000	0.000	0.000	1.000

Table 3
Optimal solution shortage function approach

	Shortage function	Optimal portfolio		$x1$	$x2$	$x3$	$x4$
		Variance	Return				
Asset 1	0.165	0.084	0.093	0.271	0.229	0.410	0.089
Asset 2	0.101	0.180	0.099	0.000	0.313	0.465	0.221
Asset 3	0.027	0.292	0.103	0.000	0.078	0.555	0.368
Asset 4	0.000	0.900	0.110	0.000	0.000	0.000	1.000

present. To formalize this idea of positive time preference in a portfolio context, we adapt a discounted temporal efficiency measure by attributing most weight to the most recent efficiency measures composing it. This is accomplished by weighting the component efficiency measures by a discount factor, denoted ξ. This time discounting parameter is assumed to remain constant over time.

Formally, a multi-horizon or temporal, discounted shortage function can therefore be defined as follows.

Definition 3.1. For all temporal portfolio paths $\mathscr{X}$, the discounted temporal shortage function is defined for all ξ such that $0 < \xi < 1$ as follows:

$$\mathscr{S}^{\xi}_{\mathscr{G}}(\mathscr{X}) = \max_{\Delta \in \mathbb{R}^T_+} \left\{ \frac{1}{T} \sum_{t=1}^{T} \xi^{T-t} \delta_t : \mathscr{F}(\mathscr{X}) + \Delta\mathscr{G} \in \mathscr{DR} \right\}.$$

For a given time horizon T, this amounts to looking for an arithmetic mean of simultaneous reductions in risks and expansions in returns into a path of direction $\mathscr{G}$ such that an observed risk-return path $\mathscr{F}(\mathscr{X})$ is projected onto the boundary of $\mathscr{DR}$. This definition proposes a weighted (discounted) temporal efficiency measure, whereby the weights are lower as one moves away from the present into the past.

The prospective use of the temporal model would require an alternative definition of the discounted temporal shortage function to start with, whereby the future is discounted relative to the present.[6] However,

[6] Indeed, one can define a prospective discounted temporal shortage function as:

$$\mathscr{S}^{\xi}_{\mathscr{G},p}(\mathscr{X}) = \max_{\Delta \in \mathbb{R}^T_+} \left\{ \frac{1}{T} \sum_{t=1}^{T} \xi^{t} \delta_t : \mathscr{F}(\mathscr{X}) + \Delta\mathscr{G} \in \mathscr{DR} \right\},$$

whereby the weights decline as one moves away from the present into the future.

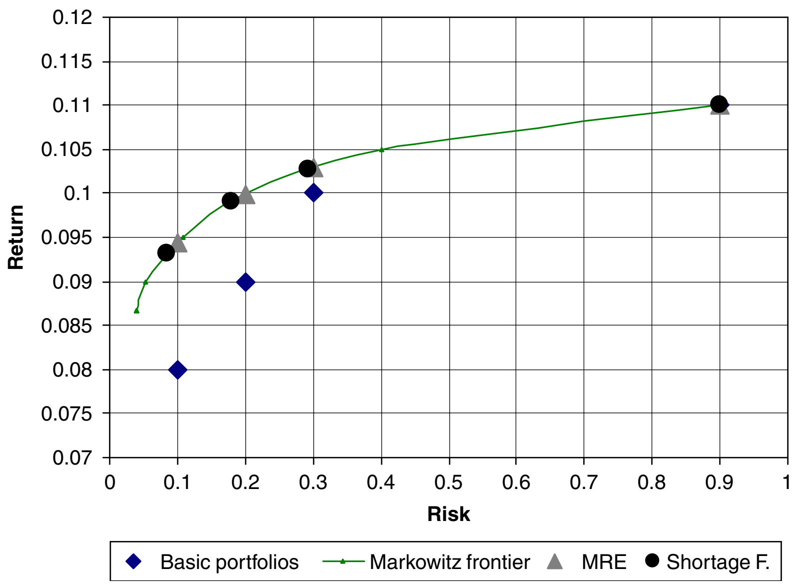

Fig. 1. Optimal portfolios: mean return augmentation versus shortage function approaches for the static numerical example.

reasons of space preclude developing this prospective approach completely in parallel with the retrospective benchmarking approach. Therefore, the remainder of this contribution focuses exclusively on the retrospective approach.

One can immediately proof the following proposition with respect to this discounted temporal shortage function.

Proposition 3.2. *For all temporal paths of portfolio* $\mathscr{X}$, *the discounted temporal shortage function can be written*:

$$\mathscr{S}^{\xi}_{\mathscr{G}}(\mathscr{X}) = \frac{1}{T}\sum_{t=1}^{T} \xi^{T-t} S^t_{g^t}(x^t).$$

Proof. The proof is similar to the one of Proposition 3 in Briec et al. [18]. □

The discounted temporal shortage function thus corresponds to the average of a series of discounted static shortage functions, one for each time period.

Since Sengupta [10], it is well-known that PE measures can be calculated relative to non-parametric frontiers. Assuming there are n assets (or funds) observed over the time period considered, an observed risk-return path for any asset is evaluated using the discounted, temporal shortage function by computing the following quadratic program:

$$\begin{aligned}
\max \quad & \frac{1}{T}\sum_{t=1}^{T} \xi^{T-t}\delta_t \\
\text{s.t.} \quad & V[R(y^t_k)] - \delta^t g^t_V \geqslant \sum_{k,l} \mathrm{Cov}[R^t_k, R^t_l] x^t_k x^t_l \\
& t = 1, \ldots, T, \\
& E[R(y^t_k)] + \delta^t g^t_E \leqslant \sum_{i=1,\ldots,n} x^t_i E[R^t_i] \quad t = 1, \ldots, T, \\
& \delta^t \geqslant 0,\ x^t_i \geqslant 0,\ i = 1, \ldots, n,\ t = 1, \ldots, T.
\end{aligned} \tag{3.2}$$

Notice three important differences with the approach advocated in Morey and Morey [11]. First, these authors ignore time discounting (i.e., $\xi = 1$). Second, there is a common efficiency measure imposed on all time periods, whereas we allow for a different efficiency measure in each time period (i.e., $\delta^t = \delta$ for $\forall t$). As a consequence, the portfolio weights are also constant over time ($x^1 = \cdots = x^T$) following the long-term investment perspective, while we allow them to vary over time to look for a periodically rebalanced benchmark over the time horizon, taking a short-run investment perspective. Third, similar to the static case in expression (2.4), one obtains their special cases of RC and MRE by a specific choice of direction vector path.

Remark also that the block-diagonal structure of the above mathematical program is a consequence of the time separability assumption, since there are no temporal linkages between the estimated portfolio problems for each period. This structure basically allows us to solve the static mathematical program for each time period separately and to compute the objective function of the above problem based on the optimal solutions of these T sub-problems at the end (see also [18] for a similar case in a production context).

Similar to the static mean-variance model, we can now define a discounted temporal mean-variance utility function over a given time horizon T:

$$\mathscr{U}_{(\rho,\mu)}(x^t) = \frac{1}{T}\sum_{t=1}^{T} \xi^{T-t} U_t(x^t), \tag{3.3}$$

where $U_t(x^t)$ stands for the static mean-variance utility at time period t (as defined in expression (2.5)). Moreover, μ and ρ are two T-dimensional nonnegative vectors. In addition, we define the corresponding indirect temporal utility function as

$$\mathscr{V}(\rho,\mu) = \max_{\mathscr{X}\in\mathfrak{I}^T} \mathscr{U}_{(\rho,\mu)}(\mathscr{X}). \tag{3.4}$$

where $\mathfrak{I}^T$ represents the set of admissible portfolios. In view of the temporal separability of $\mathscr{U}_{(\rho,\mu)}$, this temporal indirect utility function is the discounted sum of the indirect utility functions in each time period:

$$\mathscr{V}(\rho,\mu) = \frac{1}{T}\sum_{t=1}^{T} \xi^{T-t} V_t(\rho_t,\mu_t), \tag{3.5}$$

where $V_t(\rho_t,\mu_t)$ stands for the indirect utility function at each time period t, as defined in expression (2.6) (see again [18] for details).[7]

The mathematical program to compute this temporal indirect utility function can be written as follows:

$$\begin{aligned} \max \quad & \frac{1}{T}\sum_{t=1}^{T} \xi^{T-t}(\mu_t E[R^t(x^t)] - \rho_t \mathrm{Var}[R^t(x^t)]) \\ \text{s.t.} \quad & \sum_{i=1,\dots,n} x_i^t = 1, \quad x^t \geqslant 0, \quad t=1,\dots,T. \end{aligned} \tag{3.6}$$

Remark again that the block-diagonal structure of the above mathematical program allows us to solve the static mathematical program per time period and to compute the objective function of the above problem based on the optimal solutions of these T sub-problems at the end (see Proposition 4 in [18] for a similar temporal profit function case in a production context).

[7] This result is somewhat akin to Mossin [23] who already showed that the solution of a sequence of single-period mean-variance models based on expected utility maximization yields the same answer as solving a multi-period, dynamic model under a set of stringent conditions.

In line with the static decomposition of PE in Briec et al. [17], one can distinguish between overall, allocative and portfolio efficiencies in this temporal context.

Definition 3.3. Temporal OE decomposition:

(1) Temporal OE ($\mathscr{OE}$) index is the quantity:

$$\mathscr{OE}(\mathscr{X}) = \mathscr{V}(\bar\rho,\bar\mu) - \mathscr{U}_{(\bar\rho,\bar\mu)}(\mathscr{X}),$$

where $\bar\rho = \times_{t=1}^{T} \frac{\rho_t}{\mu_t g_E^t + \rho_t g_V^t}$ and $\bar\mu = \times_{t=1}^{T} \frac{\mu_t}{\mu_t g_E^t + \rho_t g_V^t}$,

(2) Temporal AE ($\mathscr{AE}$) index is the quantity:

$$\mathscr{AE}(\mathscr{X}) = \mathscr{OE}(\mathscr{X}) - \mathscr{S}_{\mathscr{G}}^{\xi}(x),$$

(3) Temporal PE ($\mathscr{PE}$) index is the quantity:

$$\mathscr{PE}(\mathscr{X}) = \mathscr{S}_{\mathscr{G}}^{\xi}(x).$$

While temporal PE guarantees that a portfolio remains at the boundary of the temporal mean-variance portfolio frontier, temporal AE ensures in addition that a portfolio maximizes the temporal indirect utility function. The notion of temporal OE ensures compliance with both of these ideals simultaneously.

A final theoretical development is a duality result between the time-discounted temporal shortage function and the temporal indirect utility function.

Proposition 3.4. *For all portfolio temporal paths $\mathscr{X}$, we have*:

$$\mathscr{S}_{\mathscr{G}}^{\xi}(\mathscr{X}) = \min_{(\rho,\mu)\in\mathbb{R}_+^{2\times T}} \{\mathscr{V}(\rho,\mu) - \mathscr{U}_{(\rho,\mu)}(\mathscr{X}) : \rho_t g_V^t + \mu_t g_E^t = 1,\ t=1,\dots,T\}.$$

Proof. See Appendix. □

This duality result confirms that our discounted, temporal shortage function does have an economic interpretation, while the Morey and Morey [11] efficiency measures are only valid when imposing extreme risk aversion parameters on the temporal direct utility function.

As a matter of fact, it is possible to reinterpret Morey and Morey [11] in our framework by computing a long-run discounted temporal shortage function. This boils down to simply adding the constraint $x^1 = \cdots = x^T$ to the quadratic program (3.2) of the short-run discounted temporal shortage function. A non-negative rebalancing bonus can then be expressed as the difference between the short- and long-run temporal discounted temporal shortage functions. It indicates the potential, gross benefits from adapting an active versus a passive portfolio management strategy. Of course, the net

Table 4
Numerical example: temporal version

	Means (%)	Variance–covariance matrix			
		Asset 1	Asset 2	Asset 3	Asset 4
Year 1					
Asset 1	8	0.1	0.03	−0.08	0.05
Asset 2	9	0.03	0.2	0.02	0.03
Asset 3	10	−0.08	0.02	0.3	0.2
Asset 4	11	0.05	0.03	0.2	0.9
Year 2					
Asset 1	8.25	0.18	0.01	−0.18	0.25
Asset 2	9.50	0.01	0.25	0.12	0.13
Asset 3	10.25	−0.18	0.12	0.37	0.22
Asset 4	11.75	0.25	0.13	0.22	0.95
Year 3					
Asset 1	9.50	0.1	0.05	−0.12	0.15
Asset 2	10.75	0.05	0.2	0.05	0.23
Asset 3	11.00	−0.12	0.05	0.3	0.25
Asset 4	12.50	0.15	0.23	0.25	0.8

benefits depend on the cost of management and transaction costs. In an analogous way, it is possible to define a long-run temporal indirect utility function. It can be computed by again simply adding the constraint $x^1 = \cdots = x^T$ to the quadratic program (3.6) for the short-run temporal indirect utility function. Again, a non-negative rebalancing bonus can be defined in terms of the difference between short- and long-term temporal indirect utility functions.

Analogous to result in Eq. (2.7), a final result is the derivation of a temporal shadow risk aversion defined by

$$\Gamma(\mathscr{X}) = \sum_{t=1}^{T} \xi^{T-t} \frac{\rho_t^*(x^t)}{\mu_t^*(x^t)} \tag{3.7}$$

and

$$\begin{aligned}(\rho^*(\mathscr{X}), \mu^*(\mathscr{X})) &= \operatorname*{arg\,min}_{(\rho,\mu)\in\mathbb{R}_+^{2\times T}} \{\mathscr{V}(\rho,\mu) - \mathscr{U}_{(\rho,\mu)}(\mathscr{X}) : \rho_t g_V^t \\ &\quad + \mu_t g_E^t = 1,\ t = 1, \ldots, T\}\end{aligned} \tag{3.8}$$

is a temporal adjusted risk aversion function. This temporal version of the static shadow risk aversion reflects the discounted average risk-aversion characterizing the observed portfolio behavior over a given time period T. It has an obvious use to trace whether the evolution of portfolio management on an average adheres to a specified risk profile and management style and its evolution over time can be followed easily by shifting the time window.

We end with a computational argument in favor of the shortage function approach. The decomposability of our approach implies a low computational cost from a practical point of view. Assuming a certain time horizon for evaluation purposes is maintained, then the Morey and Morey [11] approaches necessitate recomputing the whole mathematical program whenever new data become available and the time window is moved one unit of time across the data. In contrast, in the shortage function approach the appearance of a new period just implies computing a static mean-variance model with a static shortage function and recomputing the composite temporal objective function at the end.

3.2. Numerical example

This numerical example supposedly contains four assets observed over three years. To simplify matters, the first year coincides with the Benninga [22] example above. The second and third years are simple variations on these numbers. Table 4 contains the data for the temporal numerical example with a 3 year horizon.

We only show the optimal results for the first year in Fig. 2. Original assets and the Markowitz portfolio frontier are identical to Fig. 1. Also, the shortage function approach yields the same frontier projection. However, the mean MRE efficiency approach now falls short of the frontier. Indeed, by looking for a common factor to

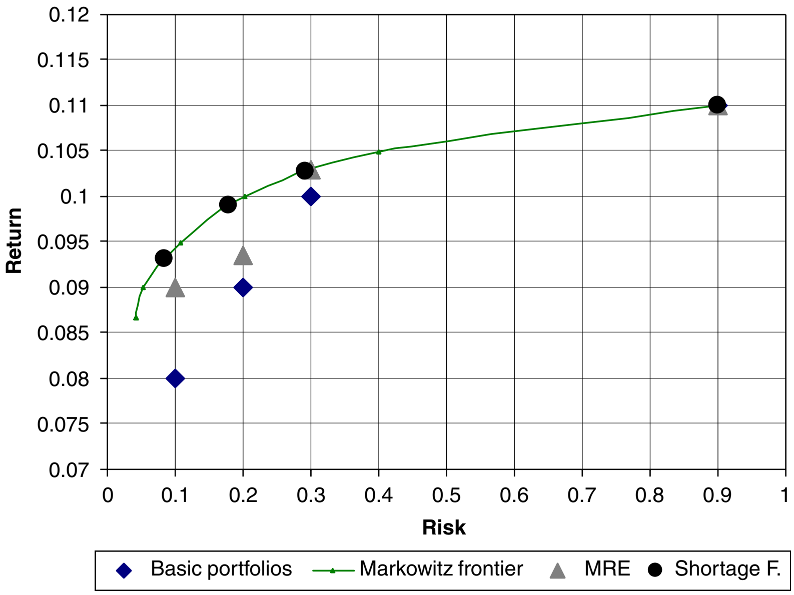

Fig. 2. Optimal portfolios: mean return augmentation versus shortage function approaches for the temporal numerical example.

augment returns for each of the years, it foregoes opportunities to augment return in any single year. Thus, by computing a common scalarwise expansion of returns, the mean MRE efficiency approach does not put the evaluated path of portfolios onto the Markowitz frontier in all single periods. In contrast, the proposed shortage function approach does project each evaluated portfolio path onto the frontier in each period.

4. Empirical illustration

To illustrate the differences between our own approach and the efficiency measures proposed by Morey and Morey [11], we use their original data set of 26 mutual funds evaluated over three time horizons: a 3, a 5 and a 10-year time period. All these mutual funds were classified as aggressive growth funds according to Morningstar. In the Morey and Morey [11] article, using monthly percentage return data one finds for each of the 3, 5 and 10-year time periods the following information: (i) mean monthly returns (see their Table 1 on p. 249), and (ii) variance–covariance matrices (see their Table 8 on pp. 256–257 (Appendix B)).

We report on the following results in Table 5. First, the mean MRE and RC efficiency measures proposed in Morey and Morey [11]. We contrast these results to the ones based on a temporal shortage function without time discounting (i.e., $\xi = 1$) and with time discounting (i.e., $\xi = 0.95$). Second, given a 5% time discounting rate, we complement the temporal shortage function with time discounting (i.e., PE) with an allocative and overall efficiency component. When computing the temporal indirect utility function, it was assumed that the risk parameters $\mu_t = 1$ and $\rho_t = 2$, which implies a degree of absolute risk aversion $\varphi_t = 2$. We only report the efficiency measures, not the portfolio weights and eventual slack variables. To make the mean MRE and RC efficiency measures comparable to the temporal shortage function, we have applied the following transformations: (i) MRE equals the original Morey and Morey [11] mean MRE efficiency score minus 1, and (ii) RC equals 1 minus the original Morey and Morey [11] RC efficiency measure.

The average performance of the mutual funds is poor. The MRE measure indicates a 23.5% potential for improvement, while the RC measure yields a 22.3% scope for risk reduction. The undiscounted shortage function is situated in between and reveals on average a 23.3% improvement in both return and risk dimensions. The Morey and Morey [11] measures agree that six mutual funds are efficient. The undiscounted shortage function confirms the efficiency of two of these observations ("AIM Aggressive Growth" and "Fund Manager Aggressive Grth"), but declares the other four observations as inefficient.

We now turn to the temporal efficiency decomposition with time discounting. On the one hand, adding a time discounting parameter reduces the average level

Table 5
Empirical temporal decomposition results

Fund name	MRE	MRC	$\mathscr{PE}^{\text{a}}$	$\mathscr{PE}^{\text{b}}$	$\mathscr{AE}^{\text{b}}$	$\mathscr{OE}^{\text{b}}$
20th Century Ultra Investors	0.000	0.000	0.373	0.175	0.344	0.519
44 Wall Street Equity	0.049	0.055	0.027	0.182	0.091	0.272
AIM Aggressive Growth	0.000	0.000	0.000	0.023	0.436	0.459
AIM Constellation	0.000	0.000	0.235	0.125	0.336	0.461
Alliance Quasar A	0.602	0.465	0.398	0.381	0.054	0.436
Delaware Trend A	0.178	0.339	0.325	0.215	0.251	0.466
Evergreen Aggressive Grth A	0.234	0.384	0.334	0.253	0.238	0.492
Founders Special	0.127	0.179	0.236	0.194	0.193	0.387
Fund Manager Aggressive Grth	0.000	0.000	0.000	0.199	0.013	0.212
IDS Strategy Aggressive B	0.292	0.276	0.329	0.332	0.051	0.383
Invesco Dynamics	0.172	0.299	0.207	0.159	0.240	0.399
Keystone Amer Omega A	0.121	0.174	0.281	0.226	0.139	0.365
Keystone Small Co Grth (S-4)	0.169	0.307	0.185	0.188	0.316	0.504
Oppenheimer Target A	0.277	0.264	0.206	0.200	0.102	0.301
Pacific Horizon Aggr Growth	0.257	0.355	0.424	0.335	0.152	0.487
PIMCo Adv Opportunity C	0.000	0.000	0.060	0.103	0.378	0.481
Putnam Voyager A	0.006	0.012	0.158	0.142	0.225	0.366
Security Ultra A	1.070	0.526	0.453	0.442	0.024	0.466
Seligman Capital A	0.332	0.311	0.346	0.297	0.086	0.383
Smith Barney Aggr Growth A	0.208	0.357	0.331	0.290	0.209	0.500
State St. Research Capital C	0.056	0.120	0.165	0.141	0.333	0.474
SteinRoe Capital Opport	0.352	0.358	0.231	0.247	0.185	0.432
USAA Aggressive Growth	0.697	0.444	0.326	0.361	0.126	0.486
Value Line Leveraged Gr Inv	0.162	0.165	0.131	0.189	0.131	0.321
Value Line Spec Situations	0.746	0.404	0.291	0.363	0.087	0.451
Winthrop Focus Aggr Growth	0.000	0.000	0.008	0.029	0.080	0.109
Average	0.235	0.223	0.233	0.223	0.185	0.408
St. deviation	0.270	0.172	0.135	0.105	0.118	0.099
Max.	1.070	0.526	0.453	0.442	0.436	0.519
Min.	0.000	0.000	0.000	0.023	0.013	0.109

[a] $\xi = 1$ (no time discounting).
[b] $\xi = 0.95$.

Table 6
Product–moment correlations between efficiency measures

	MRE	MRC	$\mathscr{PE}^{\text{a}}$	$\mathscr{PE}^{\text{b}}$	$\mathscr{AE}^{\text{b}}$	$\mathscr{OE}^{\text{b}}$
MRE	1.000	0.839	0.629	0.854	−0.528	0.279
MRC		1.000	0.747	0.850	−0.414	0.409
$\mathscr{PE}^{\text{a}}$			1.000	0.798	−0.217	0.588
$\mathscr{PE}^{\text{b}}$				1.000	−0.610	0.336
$\mathscr{AE}^{\text{b}}$						0.542
$\mathscr{OE}^{\text{b}}$						1.000

[a] $\xi = 1$ (no time discounting).
[b] $\xi = 0.95$.

of inefficiency compared to the levels measured by the undiscounted temporal shortage function. On the other hand, according to the discounted temporal shortage function not a single mutual fund is efficient. For our given choice of absolute risk aversion, temporal overall inefficiencies are substantial: on an average funds could improve its return and reduce its risk dimensions by 41%. Not a single mutual fund perfectly suits the investors preferences. Only “Winthrop Focus Aggr Growth” comes close with a temporal overall inefficiency of only about 11%. The residual degree of temporal allocative inefficiency is of about equal size as the temporal portfolio inefficiency (18.5% compared to 22.3%).

In addition to knowing the exact performance levels of each benchmarking method, it is interesting to see to which extent these methods generate the same rankings. To that purpose, we report the product–moment correlations in Table 6. The agreement among Morey and Morey [11] efficiency measures and the temporal shortage functions (both with and without time discounting) is rather high. This could indicate that the difference between adopting a short-term and a long-term investment perspective has not that great implications for the current sample of mutual funds in terms of ranking.

In contrast, the temporal allocative inefficiency and overall inefficiency components correlate well with one another, but stand apart from the rest. To some extent, this underlines the importance of the choice of risk parameters and the implied degree of absolute risk aversion.

5. Conclusion

In this contribution, we have proposed to benchmark portfolios by looking simultaneously for RC and MRE using the discounted temporal shortage function within a multi-horizon framework. This complements the work of Morey and Morey [11] that employs a common proportional factor to either contract all risk dimensions or expand all return dimensions, a perspective suitable for evaluating long-term investment. The fact that our discounted temporal shortage function is dual to a discounted temporal indirect utility function underscores the natural interpretation of this generalized efficiency measure in terms of investor's preferences. Furthermore, the fact that time discounting is traditionally applied to multi-horizon indirect utility functions implies that proper time discounting should also apply to the underlying temporal efficiency measure when adopting a short-term or long-term investment perspective. We think that the general idea of looking for both RC and MRE over a given time horizon may prove useful in a wide range of financial models.

Of course, one should be aware of the limitations of the proposed approach. These models are mainly aimed at retrospectively gauging portfolio performance over a given time horizon. This is in line with the initial idea of Morey and Morey [11] to come up with an alternative rating scheme for mutual funds relative to the rankings provided by companies like Lipper Analytical Services, Morningstar, among others. Of course, as briefly pointed out, the same models can in principle also be applied for prospective benchmarking provided adequate information on future returns is available.

A straightforward extension of the currently developed models is to include higher moments when determining optimal portfolios. For instance, as shown by Joro and Na [24] the use of efficiency measures allows us to include a preference for positive skewness in addition to the two-dimensional mean-variance model. Furthermore, these models can probably be made more useful as a planning tool for assessing optimal dynamic portfolio management in a more active portfolio management strategy by considering the following avenues for future work. First, it could be interesting to relate the long-term investment perspective implicit in the Morey and Morey [11] contribution to the expected geometric mean optimization advocated for long-term investments by Markowitz [19], Hakansson [25], and many others (see also [26] for a recent contribution). Second, apart from some other conditions, we know since Mossin [23] that single-period models are no longer good approximations to multi-period, dynamic models when temporal separability no longer holds (e.g., because returns are temporally dependent). Thus, for planning purposes one would ideally need truly dynamic portfolio optimization models in either continuous (e.g., [27]) or discrete (e.g., [28]) time. These models continue to develop in a variety of directions: from the management of fixed-income portfolios with embedded options (see, e.g., [29]) to the derivation of analytical optimal solutions under specific conditions (e.g., [30]). It remains an open question whether efficiency measures could be integrated into these type of models to evaluate and compare alternative future portfolio paths.

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Appendix

Proof of Proposition 3.4. We have shown that:

$$\mathscr{S}^{\xi}_{\mathscr{G}}(\mathscr{X}) = \frac{1}{T}\sum_{t=1}^{T} \xi^{T-t} S^t_{g^t}(x^t).$$

From Briec et al. [17], at each time period the shortage function satisfies the dual relationship:

$$S^t_{g^t}(x^t) = \min_{(\rho_t,\mu_t)\in\mathbb{R}^2_+} \{V_t(\rho_t,\mu_t) - U_{(\rho_t,\mu_t)}(x^t) : \rho_t g^t_V + \mu_t g^t_E = 1\}.$$

Hence,

$$\mathscr{S}_{\mathscr{G}}^{\xi}(\mathscr{X}) = \frac{1}{T}\sum_{t=1}^{T}\xi^{T-t} \times \min_{(\rho_t,\mu_t)\in\mathbb{R}_+^2}\{V_t(\rho_t,\mu_t) - U_{(\rho_t,\mu_t)}(x^t)\colon \rho_t g_V^t + \mu_t g_E^t = 1\}.$$

The temporal separability of this optimization program yields:

$$\mathscr{S}_{\mathscr{G}}^{\xi}(\mathscr{X}) = \min_{\substack{(\rho_t,\mu_t)\in\mathbb{R}_+^2\\ \rho_t g_V^t+\mu_t g_E^t=1, t=1,\dots,T}} \left\{\frac{1}{T}\sum_{t=1}^{T}\xi^{T-t}V_t(\rho_t,\mu_t) - \frac{1}{T}\sum_{t=1}^{T}\xi^{T-t}U_{(\rho_t,\mu_t)}(x^t)\right\},$$

which ends the proof. □

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