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Mean-Variance-Skewness Portfolio Performance Gauging: A General Shortage Function and Dual Approach

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This paper proposes a nonparametric efficiency measurement approach for the static portfolio selection problem in mean-variance-skewness space. A shortage function is defined that looks for possible increases in return and skewness and decreases in variance. Global optimality is guaranteed for the resulting optimal portfolios. We also establish a link to a proper indirect mean-variance-skewness utility function. For computational reasons, the optimal portfolios resulting from this dual approach are only locally optimal. This framework permits to differentiate between portfolio efficiency and allocative efficiency, and a convexity efficiency component related to the difference between the primal, nonconvex approach and the dual, convex approach. Furthermore, in principle, information can be retrieved about the revealed risk aversion and prudence of investors. An empirical section on a small sample of assets serves as an illustration.

Key words: shortage function; efficient frontier; mean-variance-skewness portfolios; risk aversion; prudence
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1. Introduction

The seminal work of Markowitz (1952) in modern portfolio theory trades off the risk and expected return of a portfolio in a static context. Portfolios whose expected return cannot increase unless their risk increases define an efficient frontier, i.e., a Pareto-optimal subset of portfolios. His work maintains strong assumptions on probability distributions and Von Neumann-Morgenstern utility functions. Furthermore, the computational cost of solving quadratic programs in these days led Sharpe (1963) to propose a simpler "diagonal" model and inspired Sharpe (1964) and Lintner (1965) to develop the capital asset pricing model (CAPM), an equilibrium model assuming that all agents have similar expectations about the market. Widespread tools for gauging portfolio efficiency, such as Sharpe (1966) and Treynor (1965) ratios and Jensen (1968) alpha, have mainly been developed with reference to these developments, and in particular, CAPM. Despite these and later enhancements, the Markowitz model still offers the most general framework.

The main theoretical difficulty with the so-called parametric approach where utility depends on the first and second moments (i.e., mean and variance) of the random variable's distribution is that it is only consistent with expected utility and its von Neumann-Morgenstern axioms of choice when (i) asset processes are normally distributed (hence, higher moments can be ignored), or (ii) investors have quadratic utility functions (e.g., Samuelson 1967). However, a plethora of empirical studies shows that portfolio returns are generally not normally distributed. Furthermore, investors prefer positive skewness because it implies a low probability of obtaining a large negative return. In particular, the observation that increased diversification leads to skewness loss and the widespread phenomenon of imperfectly diversified portfolios may well reveal a preference for positive skewness among investors, rather than simply capital market imperfections (Kraus and Litzenberger 1976, Simkowitz and Beedles 1978, Kane 1982). Theoretically, positive skewness preference is related to the positivity of the third derivative of the utility function: the prudence notion is to marginal utility what risk aversion is to utility.[1] Furthermore, ever

[1] As Kimball (1990, p. 54) states: "The term 'prudence' is meant to suggest the propensity to prepare and forearm oneself in the face of uncertainty, in contrast to 'risk aversion,' which is how much

since Samuelson (1970) it is known that the mean-variance (MV) approach is adequate when return distributions are compact and when portfolio decisions are made frequently (almost continuously) such that the risk parameter becomes sufficiently small. However, when the portfolio decision is limited to a finite time interval and rebalancing is restricted, then higher moments (cubic utility and beyond) are needed because the quadratic approximation is not locally of high contact.

While the limits of a quadratic approximation of the utility function are acknowledged, the development of third- or higher-degree polynomial forms for the utility function as part of operational procedures for constructing portfolios has been hampered mainly by computational problems (see Markowitz 1991).[2] Several alternative criteria for portfolio selection based on higher-order moments have been developed (Philippatos 1979, Wang and Xia 2002), but so far not a single generally valid procedure seems to have emerged. It is possible to distinguish between primal and dual approaches to determine mean-variance-skewness (MVS) portfolio frontiers. An example of the primal approach is found in Lai (1991) and Wang and Xia (2002), who determine MVS portfolios via a multiobjective programming approach.[3] In line with the work of Farrar (1962) in the basic Markowitz model, the dual approach starts from a specification of the indirect MVS utility function and determines optimal portfolios via its parameters reflecting preferences for risk and skewness (see, e.g., Jondeau and Rockinger 2006, Harvey et al. 2003 for recent studies). In the current state of affairs, however, there is no connection between primal and dual approaches.

More in general, as the dimensionality of the portfolio selection problem increases, it becomes more difficult to develop a geometric interpretation of the portfolio frontier and to select a most preferred portfolio among its boundary points. While the geometric construction of an MV portfolio frontier is trivial, no general procedure currently exists to generate a three-dimensional geometric representation of the MVS portfolio frontier. Even if one could come up with such a procedure, it would obviously be of no help for higher dimensions when approximating higher-order polynomial forms of the expected utility function.

It is our basic contention that a general procedure to describe the boundary of the set of portfolios and to pick a point among these boundary points in terms of risk preferences requires the use of a distance function. In consumer theory, the distance function is employed to position bundles of goods with respect to a target utility level of the utility function, and this distance function turns out to be dual to the expenditure function (e.g., Deaton 1979). In production theory, Luenberger (1995) introduced the shortage function as a distance function that simultaneously looks for reductions in inputs and expansions in outputs and that is dual to the profit function.[4] Thus, a distance (gauge) function offers a perfect representation of multidimensional choice sets and can position any point relative to the boundary (frontier) of the choice set. Because points beneath the frontier are in general inefficient, distance functions have an interpretation as indicators of inefficiency. Obviously, points on the frontier of a choice set are efficient. Furthermore, thanks to their duality relationships, one can select among the efficient boundary points a point that optimizes an economically meaningful objective function.

Briec et al. (2004) integrate the shortage function as a representation of the MV space and as an efficiency measure into the Markowitz model. They also develop a dual framework to assess the degree of satisfaction of investors' preferences (see, e.g., Farrar 1962). They propose a decomposition of portfolio performance into allocative and portfolio efficiency.[5] Moreover, via the shadow prices associated with the shortage function, duality yields information about investors' risk aversion.[6]

In this paper, the shortage function is extended to the MVS space to account for a preference for positive skewness in addition to a preference for returns and an aversion to risk. The shortage function projects any (in)efficient portfolio exactly on the three-dimensional MVS portfolio frontier. Anticipating a major result, we prove that the shortage function achieves a global optimal solution on the boundary of the nonconvex MVS portfolio frontier. Starting from a sample of

one dislikes uncertainty and would turn away from uncertainty if possible."

[2] While a cubic utility function need not guarantee decreasing absolute risk aversion everywhere, it is already more satisfactory than a quadratic utility function which implies increasing absolute risk aversion for all wealth levels, a counterintuitive assumption.

[3] The goal-programming model of Lai (1991) is by far the most popular in empirical studies: it has been applied to 14 major stock markets by Chunhachinda et al. (1997), to an extended set of 17 stock markets in Prakash et al. (2003), and to Japanese and U.S. stocks by Sun and Yan (2003), among others.

[4] This shortage function generalizes the input distance function, that is dual to the cost function, and the output distance function, that is dual to the revenue function.

[5] This work generalizes earlier contributions transposing efficiency measures from production theory into basic portfolio analysis: for instance, Morey and Morey (1999) focus either on return expansion or on risk reduction, but ignore that in general investors may be assumed to prefer higher returns and reduced risk simultaneously.

[6] This procedure has the additional advantage of being simple compared to other nonparametric estimators (see, for instance, Lien 2002 and the references cited therein).

observed portfolios with unknown efficiency status, this shortage function projects a portfolio for which improvements can be found, in terms of increasing return and skew and decreasing risk, onto the MVS frontier and labels these inefficient. By contrast, when no such improvements can be found, then the initial portfolio must have been part of the MVS frontier right at the outset and it obtains the label efficient. Proceeding in this way, the shortage function reconstructs parts of the unknown MVS portfolio frontier. Just like in the MV case, all points on the MVS portfolio frontier are Pareto efficient. Furthermore, to choose among these frontier portfolios, we develop a dual approach specifying an MVS utility function. For given risk aversion and prudence parameters, we can pick an optimal point on the boundary of the nonconvex MVS portfolio frontier. Furthermore, by proving a duality result between the shortage function and the indirect MVS utility function, we show that our shortage function approach is not devoid of economic interpretation, but rather that both approaches are firmly integrated.

In general, the shortage function accomplishes four goals of both theoretical and practical importance: (i) it rates portfolio performance by measuring a distance between a portfolio and its optimal benchmark projection onto the primal MVS efficient frontier; (ii) it provides a nonparametric estimation of an inner bound of the true but unknown portfolio frontier; (iii) it judges simultaneously return and skewness expansions and risk contractions; and (iv) it provides a new, dual interpretation of this portfolio efficiency distance. To expand on the latter possibility, thanks to the above mentioned duality result the shortage function can under some specific conditions reveal via its shadow prices the (shadow) risk aversion and prudence compatible with the projection of an inefficient portfolio at the frontier.

This framework based on the shortage function improves on various attempts to determine MVS portfolio frontiers.[7] First looking at the primal approach, the estimation of MVS portfolio frontiers via the multiobjective programming problem does not comply with the theoretical notion of a frontier portfolio. Minimizing deviations from three objectives simultaneously only guarantees a solution "close" to the frontier. Furthermore, there is no clear performance measure and there is no link whatsoever between the parameters weighting the deviations from the three moment objectives and the parameters of the expected utility function. By contrast, the use of distance functions avoids any compromise between the three objectives, provides a clear performance measure, and is via duality firmly linked with risk preferences. Furthermore, there are a series of primal contributions that tend to solve the MVS portfolio problem by privileging one or two of the objectives at the cost of the other(s). Konno and Suzuki (1995) trace the MVS portfolio frontier by maximizing skewness, and focus thereby on finding approximate optimal solutions using piecewise linear approximations of nonlinear objective function and constraints. Adopting the efficiency measures proposed in Morey and Morey (1999), Joro and Na (2006) determine MVS portfolio frontiers by minimizing the risk reduction for a given MVS portfolio. Athayde and Flôres (2004) look for the analytical solution characterizing the MVS portfolio frontier assuming a risk-free asset and shorting, whereby the objective is to minimize the variance for given mean and skewness. Womersley and Lau (1996) maximize the skewness divided by the standard deviation cubed, assuming that maximizing the third moment tends to minimize the variance.

While all approaches are capable to determine some Pareto efficient points on the MVS frontier (with a qualification perhaps for the multiobjective programming approach), these primal approaches are disconnected from any preference information eventually allowing to select one portfolio among those on the Pareto efficient MVS frontier. In fact, it is shown below that most of these approaches can be re-interpreted as special cases of our shortage function, whereby the direction of projecting onto the frontier privileges one of the three dimensions: e.g., one only looks for improvements in skewness. Therefore, one should realize that some of these methods may lead to points on the unknown MVS frontier that are probably unattractive from the viewpoint of general investor preferences. By contrast, our approach caters for more general investor preferences in that we seek simultaneously improvements in return and skewness and reductions in risk. Furthermore, our approach is more general in that we impose the weakest of possible assumptions. For instance, we ignore the presence of a risk-free asset as well as the possibility of shorting.

Current dual approaches are hampered by a lack of knowledge of preferences for risk and skewness (e.g., Jondeau and Rockinger 2006, Harvey et al. 2003) and suffer from their lack of integration with primal approaches. Because the MVS portfolio frontier is nonconvex, the optimization of an indirect utility function in the dual approach only ensures local optimal solutions from a computational point of view. This inherent characteristic of the MVS decision problem can only be remedied via the development of global optimization algorithms. Furthermore,

[7] Early attempts looking at skewness but ignoring co-skewness (e.g., Arditti and Levy 1975) are disregarded.

it inevitably convexifies part of the underlying nonconvex MVS portfolio frontier. This may carry the risk that certain target portfolios based on particular specifications of the utility function are infeasible in practice. But, our shortage function approach is compatible with general investor preferences and selects optimal portfolios without assuming a detailed knowledge on the preference parameters defining the indirect utility function. Furthermore, it can in the long run contribute to a better understanding of risk preferences via its estimation of (shadow) risk aversion and prudence. This is a major advantage of opting for a micro-economic tool like the shortage function integrating primal and dual approaches.

The limited experience with MVS portfolio selection established so far shows that the composition of an optimal MVS portfolio differs from the MV portfolio, and that the resulting return (risk) may well be lower (higher) in trade-off with a higher positive skewness that is achieved (see Lai 1991, Prakash et al. 2003, among others). Sun and Yan (2003) make the observation that while many studies indicate that ex post stock returns are positively skewed, most of them find skewness to be persistent only for individual stocks not for portfolios (e.g., Simkowitz and Beedles 1978). However, these studies do not start from MVS efficient portfolios. These authors show that taking skewness preference seriously and using the Lai (1991) goal programming method of selecting MVS efficient portfolios for U.S. and Japanese stocks guarantees skewness persistence over time. If their results are corroborated, this implies that even ex post skewness could be used as a crude proxy for ex ante skewness when selecting optimal MVS efficient portfolios to guarantee skewness persistence.

While limiting ourselves to the three-dimensional MVS space, this contribution paves the way to any portfolio selection approach using a higher-order Taylor expansion of the utility function, as ideally dictated by the number of statistical moments that turn out to count in explaining asset prices. Finally, the interest of this approach based on cubic (nonlinear) programming concerns not only the MVS model with short sales excluded. This nonlinear programming approach remains valid as a general framework for any other traditional portfolio extension (e.g., buy-in thresholds for assets, cardinality constraints restricting the number of assets, transaction round lot restrictions, dedicated cash flow streams, immunization strategies, etc.; see Jobst et al. 2001).

The rest of this paper is organized as follows. Section 2 lays down the foundations of the analysis. Section 3 introduces the shortage function and studies its axiomatic properties. Section 4 studies the link between the shortage function and the direct and indirect MVS utility functions. A simple empirical illustration using a small sample of 35 assets (all part of the French CAC40 index) is provided in §5. Conclusions and possible extensions are formulated in §6.

2. Portfolio and Efficient Frontier: Definitions

To develop some basic definitions, consider the problem of selecting a portfolio (or fund of funds) from n financial assets (or funds). Assets are characterized by an expected return $\mathrm{E}[R_i]$ for $i \in \{1, \ldots, n\}$, by a covariance matrix $\Omega_{i,j} = \mathrm{Cov}[R_i, R_j]$ for $i, j \in \{1, \ldots, n\}$, and by a co-skewness matrix

$$\mathrm{CSK}_{i,j,k} = \mathrm{E}\big[(R_i - \mathrm{E}[R_i])(R_j - \mathrm{E}[R_j])(R_k - \mathrm{E}[R_k])\big]$$

for $i, j, k \in \{1, \ldots, n\}$.[8] Following Athayde and Flôres (2004), we transform the $n \times n \times n$ CSK matrix into a useful $n \times n^2$ matrix Λ by slicing each $n \times n$ layer and pasting them in the same order.

A portfolio $x = (x_1, \ldots, x_n)$ is composed by a proportion of each of these n financial assets ($\sum_{i=1,\ldots,n} x_i = 1$). When short sales are excluded, the condition $x_i \geq 0$ is imposed. In general, the set of admissible portfolios can be written as follows:[9]

$$\Im = \left\{ x \in \mathbb{R}^n;\ \sum_{i=1,\ldots,n} x_i = 1,\ x \geq 0 \right\}. \tag{1}$$

It is assumed throughout this paper that $\Im \neq \varnothing$.

The return of portfolio x is given by $R(x) = \sum_{i=1,\ldots,n} x_i R_i$. The expected return, its variance, and its skewness can be calculated as follows:

$$\mathrm{E}[R(x)] = \mu(x) = \sum_{i=1,\ldots,n} x_i\, \mathrm{E}[R_i] = x' M, \tag{2}$$

$$\mathrm{Var}[R(x)] = \mathrm{E}[(R(x) - \mu(x))^2] = \sum_{i,j} x_i x_j \mathrm{Cov}[R_i, R_j] = x' \Omega x, \tag{3}$$

$$\mathrm{Sk}[R(x)]$$

$$= \mathrm{E}[(R(x) - \mu(x))^3] \tag{4}$$

$$= \sum_{i,j,k} x_i x_j x_k\, \mathrm{E}[(R_i - \mu(x))(R_j - \mu(x))(R_k - \mu(x))] \tag{5}$$

$$= x' \Lambda (x \otimes x), \tag{6}$$

where $\Lambda = \mathrm{E}[(R_i - \mu(x))(R_j - \mu(x))' \otimes (R_k - \mu(x))']$ has dimension (n, n^2) to maintain a standard matrix

[8] In line with Chunhachinda et al. (1997) and Lai (1991), among others, skewness and co-skewness are defined in terms of central moments. Other definitions are available, but the choice of definition does not affect our basic results.

[9] When investors face additional constraints (e.g., transaction costs or upper limits on any fraction invested) that can be written as constraints that are linear functions of asset weights, then the set of admissible portfolios can be easily adapted (Pogue 1970, Rudd and Rosenberg 1979). See Briec et al. (2004) for this development in a similar context.

format and $\otimes$ stands for the Kronecker product. Because of certain symmetries, not all elements of these matrices need to be computed. Indeed, while the variance-covariance matrix has dimension (n, n), only

$$\binom{n+1}{2} = (n+1)n/2$$

of its elements must be computed. Similarly, while the skewness–co-skewness matrix has dimension (n, n, n), only

$$\binom{n+2}{3} = (n+2)(n+1)n/6$$

are independent (see Athayde and Flôres 2004, p. 1338).

We introduce the function $\Phi: \Im \to \mathbb{R}^3$ defined by

$$\Phi(x) = (\mathrm{E}[R(x)], \mathrm{Var}[R(x)], \mathrm{Sk}[R(x)])$$

to represent, for a given portfolio x, its expected return, variance, and skewness.

It is useful to define the MVS representation of the set $\Im$ of portfolios as the range of Φ on $\Im$:

$$\aleph = \{\Phi(x);\ x \in \Im\}. \tag{7}$$

The above set can be extended by defining an MVS (portfolio) *disposal representation set* through

$$\mathcal{DR} = \aleph + (\mathbb{R}_+ \times (-\mathbb{R}_+) \times \mathbb{R}_+). \tag{8}$$

The disposal representation set $\mathcal{DR}$ can be rewritten as follows:

$$\mathcal{DR} = \big\{(E, V, S) \in \mathbb{R}^3;\ \exists x \in \Im,\ (E, -V, S) \le (\mathrm{E}[R(x)], -\mathrm{Var}[R(x)], \mathrm{Sk}[R(x)])\big\}. \tag{9}$$

The addition of the cone is necessary for the definition of a sort of "free disposal hull" of the MVS representation of feasible portfolios.

To measure portfolio efficiency, it is necessary to define a subset of this representation set known as the efficient frontier:

Definition 2.1. In the MVS space, the weakly efficient frontier is defined as

$$\partial^M(\Im) = \big\{(E, V, S);\ (-E', V', -S') < (-E, V, -S) \Rightarrow (E', V', S') \notin \mathcal{DR}\big\}.$$

From the above definition, the weakly efficient frontier is the set of all the MVS points that are not strictly dominated in the three-dimensional space.[10]

The above definition allows defining the set of weakly efficient portfolios:

[10] It is also possible to define a strongly efficient frontier, but the above formulation simplifies most results in this contribution.

Definition 2.2. The set of weakly efficient portfolios is defined, in the simplex, as

$$\Theta^M(\Im) = \{x \in \Im;\ \Phi(x) \in \partial^M(\Im)\}.$$

By analogy to its role in the MV approach (see Briec et al. 2004), the next section introduces the shortage function (Luenberger 1995) as a performance indicator for the MVS portfolio optimization problem.

3. Shortage Function and the Frontier of Efficient Portfolios

In production theory, the shortage function measures (intuitively stated) the distance between some point of the production possibility set and the Pareto frontier (Luenberger 1995). The basic properties of the subset $\mathcal{DR}$ on which the shortage function is defined are discussed in Briec et al. (2004) in the setting of MV portfolio theory. It is now possible to extend their definition to obtain an efficiency measure in the specific context of MVS portfolio selection. Therefore, the shortage function is introduced and its properties are studied in the context of MVS portfolio theory.

Definition 3.1. Let $g = (g_E, -g_V, g_S) \in \mathbb{R}_+ \times (-\mathbb{R}_+) \times \mathbb{R}_+$. The function $S_g: \Im \to \mathbb{R}_+$ defined as $S_g(x) = \sup\{\delta;\ \Phi(x) + \delta g \in \mathcal{DR}\}$ is the shortage function for portfolio x in the direction of vector g.

The pertinence of this shortage function as a portfolio management efficiency indicator stems from its elementary properties. Because these properties can be immediately transposed from the MV into the MVS space, these properties are stated without extensive comments and proof.

Proposition 3.2. *S_g satisfies the following properties*:

(a) *If $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$, then $S_g(x) = 0 \Leftrightarrow x \in \Theta^M(\Im)$ (weak efficiency).*

(b) *S_g is MVS weakly monotonic, i.e.,*

$$\big(\mathrm{E}[R(x')], -\mathrm{Var}[R(x')], \mathrm{Sk}[R(x')]\big) \le \big(\mathrm{E}[R(x)], -\mathrm{Var}[R(x)], \mathrm{Sk}[R(x)]\big)$$

implies that $0 \le S_g(x) \le S_g(x')$.

(c) *If $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$, then S_g is continuous.*

When the shortage function equals zero, the portfolio is part of the weakly efficient frontier. This only guarantees weak efficiency because it does not exclude projections on vertical or horizontal parts of the nonconvex frontier allowing for additional improvements (see expression (8)). In addition, a portfolio that is weakly dominated in terms of its return, risk, and skewness characteristics is classified as weakly less efficient. Note that the condition $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$ is not necessary in this case to guarantee weak monotonicity. Finally, this shortage function is continuous when the direction vector g is strictly positive.

The representation set $\mathcal{DR}$, defined by expression (9), can be directly used to compute the shortage function by standard cubic optimization methods. Assume a sample of m portfolios (or investment funds) $x^1, x^2, \ldots, x^m$. Now, consider a specific portfolio y^k for $x^1, x^2, \ldots, x^m$ whose performance needs to be gauged. The shortage function for this portfolio y^k under evaluation ($S_g(y^k)$) is computed by solving the following cubic program:

$$\begin{aligned} \max\ & \delta \\ \text{s.t. } & \mathrm{E}[R(y^k)] + \delta g_E \le \mathrm{E}[R(x)], \\ & \mathrm{Var}[R(y^k)] - \delta g_V \ge \mathrm{Var}[R(x)], \\ & \mathrm{Sk}[R(y^k)] + \delta g_S \le \mathrm{Sk}[R(x)], \\ & \sum_{i=1,\ldots,n} x_i = 1, \quad x_i \ge 0,\ i = 1, \ldots, n. \end{aligned} \tag{P$_1$}$$

Making use of equations (2)–(4), program (P_1) is rewritten as follows:

$$\begin{aligned} \max\ & \delta \\ \text{s.t. } & \mathrm{E}[R(y^k)] + \delta g_E \le \sum_{i=1,\ldots,n} x_i\, \mathrm{E}[R_i], \\ & \mathrm{Var}[R(y^k)] - \delta g_V \ge \sum_{i,j} \Omega_{i,j} x_i x_j, \\ & \mathrm{Sk}[R(y^k)] + \delta g_S \le \sum_{i,j,k} \mathrm{CSK}_{i,j,k} x_i x_j x_k, \\ & \sum_{i=1}^{n} x_i = 1, \quad x_i \ge 0, \quad i = 1, \ldots, n. \end{aligned} \tag{P$_2$}$$

Thus, gauging the performance of a sample of m portfolios requires computing one cubic program for each of these m portfolios in turn. Indeed, the logic is that each observation is positioned with respect to the boundary of the choice set with the help of the shortage function. All possible combinations of returns, risk, and skewness of the portfolios in the sample that can be combined to constitute the MVS portfolio frontier are situated on the right-hand side (rhs) of (P_2). In turn, an evaluated portfolio is represented on the left-hand side (lhs) of (P_2): by maximizing δ, one attempts to augment its return and skewness and reduce its risk in the direction of vector g. If $\delta = 0$, then the evaluated portfolio is efficient and part of the boundary. Otherwise, there exists a combination of other portfolios that yields a higher return and skewness and a lower risk; the evaluated portfolio is situated below the boundary, thus inefficient.

In addition to existing portfolios, it is also possible to evaluate fictitious portfolios. In that case, one simply fills out the target values for return, risk, and skewness one would be eager to achieve in the lhs, and program (P_2) computes whether there is a combination of portfolios in the sample that could generate these values or improve on them. If the target values happen to lay on the nonconvex portfolio frontier, then the optimal delta equals zero. In the more likely event that these target values are situated below the frontier, delta is positive. When the target values cannot be generated from the current sample, then (P_2) is simply infeasible (the target values are "outside" the portfolio frontier). Theoretically, one could in this way define a grid of target values as a starting point to find a series of projection points on the portfolio frontier. With a sufficiently fine grid, this could allow to draw a three-dimensional geometrical representation of the primal nonconvex portfolio frontier. As indicated before, this procedure is only relevant up to the three-dimensional MVS portfolio space and its computational feasibility and practical relevance remain to be explored.

Note that, as mentioned before, the rhs of the constraints with the variance-covariance matrix and the skewness–co-skewness matrix can be rewritten to exploit all symmetries.[11] Note furthermore that dropping the third constraint leads to computing a shortage function relative to the MV model (Briec et al. 2004).

The above programs are special cases of the following standard, nonlinear (cubic) program:

$$\begin{aligned} \min\ & c^T z \\ \text{s.t. } & L_j(z) \le \alpha_j, \quad j = 1 \ldots q, \\ & Q_k(z) \le \beta_k, \quad k = 1 \ldots r, \\ & N_l(z) \le \gamma_l, \quad l = 1 \ldots t, \\ & z \in \mathbb{R}^p, \end{aligned} \tag{P$_3$}$$

where L_j is a linear map for $j = 1 \ldots q$, Q_k is a positive semidefinite quadratic form for $k = 1 \ldots r$, and N_l is cubic form for $l = 1 \ldots t$. In the case of program (P_2), $p = n$ and $q = r = t = 1$. Program (P_3) is not a standard convex nonlinear optimization problem (see Fiacco and McGormick 1968, Luenberger 1984).

Due to this nonconvex nature, we need to state a sufficient condition showing that a local optimal solution is also a global optimal solution. The next proposition clearly demonstrates a condition such that the shortage function achieves a global optimum for the cubic program (P_2).

Proposition 3.3. *Assume that x^* is not a strict local maximum of the skewness on $\mathfrak{I}$. If (δ^*, x^*) is a local optimum of* (P_2)*, then it is a global solution. Therefore, if the first-order and second-order Kuhn-Tucker conditions hold at point (δ^*, x^*), then (δ^*, x^*) is a global maximum of* (P_2).

[11] To be explicit, the rhs of the second constraint can be rewritten as follows: $\sum_{i,j} \Omega_{i,j} x_i x_j = \sum_{i=j} \Omega_{i,j} (x_i)^2 + 2 \sum_{i<j} \Omega_{i,j} x_i x_j$, while the rhs of the third constraint can be rewritten as $\sum_{i,j,k} \mathrm{CSK}_{i,j,k} x_i x_j x_k = \sum_{i=j=k} \mathrm{CSK}_{i,j,k} (x_i)^3 + 3 \sum_{i=j,\, j\ne k} \mathrm{CSK}_{i,j,k} (x_i)^2 x_k + 6 \sum_{i<j,\, j<k} \mathrm{CSK}_{i,j,k} \cdot x_i x_j x_k$.

Proof. See the online appendix (provided in the e-companion).[12]

This proposition clearly makes our approach stand out compared to existing primal approaches listed in the introduction only guaranteeing a local optimal solution. Thus, the shortage function offers the only tool known so far providing a global optimum for the MVS portfolio approach.

Remark 3.4. S_g encompasses all existing primal proposals mentioned in the introduction. In particular, setting two subvectors of the direction vector g equal to zero generates the following possibilities: (a) $g = (g_E, 0, 0)$ yields a return maximization model; (b) $g = (0, -g_V, 0)$ yields a risk minimization model; and (c) $g = (0, 0, g_S)$ yields a skewness maximization model. Other special cases can be imagined by setting only one subvector equal to zero rather than two.

For instance, the approach of Athayde and Flôres (2004) characterizes analytically the MVS portfolio frontier by minimizing the variance (thus, in our approach it coincides with part (b)), apart from them allowing for a risk-free asset and shorting. As another series of examples, Joro and Na (2006) use a special efficiency measure that solely minimizes portfolio risk (thus, coinciding with part (b)), while Konno and Suzuki (1995) focus on maximizing skewness (thus, coinciding with part (c)).

Note that in these special cases, $(g_E, g_V, g_S) \not> 0$. Hence, there is no guarantee that the shortage function characterizes a weakly efficient portfolio. While the fact that the shortage function equaling zero only guarantees a weakly efficient portfolio is true in general, it is intuitively clear that setting part of the direction vector g equal to zero increases the chances of projecting onto vertical or horizontal parts of the nonconvex frontier.

Remark 3.5. S_g defined on the MVS space is smaller or equal to S_g defined relative to the MV space.

This remark describes a simple consequence of adding a constraint to a maximal value function: program (P_2) contains one more constraint, namely, the third skewness–co-skewness constraint, compared to the MV model using a similar shortage function. It can provide a basis for developing statistical tests for the relevance of including additional moments in the approximation of the expected utility function.[13]

The result in the previous Proposition 3.3 has an immediate consequence in terms of Kuhn-Tucker optimality, complementary slackness, and second-order conditions. As can be shown in Corollary 3.6 (see the online appendix), these otherwise local conditions become global conditions thanks to this proposition.

The next section studies the shortage function from a duality standpoint.

4. Mean-Variance-Skewness Utility and Duality: Shadow Risk Aversion and Prudence

4.1. Motivation

To show how the shortage function is linked to the dual approach based on the specification of an MVS utility function, we must establish a duality result between the shortage function and the MVS utility function. However, this cannot be done straightforwardly because the MVS portfolio frontier is nonconvex. One can only establish a duality result after convexifying this MVS portfolio frontier. Therefore, we can define another shortage function relative to this convexified MVS portfolio frontier and establish a duality result between this new shortage function and the MVS utility function. This duality result indicates that this new shortage function has an economic interpretation, which transposes to the initial shortage function.

This duality result also leads to the definition of an efficiency decomposition. Similar to standard microeconomic approaches in production and consumption theory, we distinguish principally between portfolio efficiency, the distance from the interior to the boundary of the primal MVS portfolio frontier, and allocative efficiency, the deviation from an eventual boundary portfolio to the most preferred portfolio based on some specification of an MVS utility function. Finally, we can determine the conditions under which shadow prices of the initial shortage function are valid in the sense that they convey information about the supporting MVS utility function at the tangency point. This happens when these shadow prices coincide to the shadow prices obtainable from the shortage function defined relative to the convexified MVS portfolio frontier. Basically, this presupposes that the initial shortage function projects onto a "convex" part of the primal MVS portfolio frontier. Otherwise, one cannot attribute any economic significance to these shadow prices.

These developments serve to establish a connection between our new primal approach and existing dual approaches. They also reveal that the shortage function approach has a meaningful economic interpretation. Therefore, this section is structured as follows. In the next subsection, we define an MVS utility function as a third-order polynomial approximation of expected utility. Then, we define another shortage function relative to a convexified MVS portfolio

[12] The e-companion to this paper, which is part of the online version, is available at http://mansci.pubs.informs.org/.

[13] It is possible to improve the small sample error of our nonparametric frontier estimator using either information on its asymptotic distribution of efficiency estimates, or by simulated (bootstrapped) empirical distributions (see Simar and Wilson 2000).

frontier and establish a duality result between it and the MVS utility function. We also define an efficiency decomposition. Finally, we study the properties of the shortage function that assume differentiability at the point where the function is evaluated in an effort to determine the conditions for the validity of shadow prices.

4.2. Mean-Variance-Skewness Utility Functions: Definition

From the outset, portfolio selection was conceived as a two-step procedure: the determination of the efficient set of portfolios is just the first step, preparing the selection of an optimal portfolio for a given preference structure. To provide a dual interpretation of the shortage function, a corresponding indirect utility function must first be defined.

Let the MV utility function be defined by $U_{(\mu,\rho)}(x) = \mu\,\mathrm{E}[R(x)] - \rho\mathrm{Var}[R(x)]$. This utility function satisfies positive marginal utility of expected return, and negative marginal utility of risk.

To determine an optimal portfolio corresponding to a given degree of risk aversion within the MV approach, Markowitz (1959) formalizes a quadratic optimization program maximizing the above MV utility function:

$$\begin{aligned} \max\ & \mathrm{E}[R(x)] - \varphi\mathrm{Var}[R(x)] \\ \text{s.t.}\ & \sum_{i=1\ldots n} x_i = 1, \quad x \geq 0, \end{aligned} \tag{10}$$

with $\varphi = \rho/\mu \in [0, +\infty]$ representing the degree of absolute risk aversion.

To integrate the skewness, we propose another optimization program that determines the optimal portfolio corresponding simultaneously to a given degree of risk aversion and prudence. The concept of prudence, introduced by Kimball (1990), is related to skewness preference. In combination with risk aversion, it allows to handle simultaneously variance and skewness. Kimball (1990) develops the relationship between prudence and the third derivative of the utility function in a consumption-savings approach. Since at least Kraus and Litzenberger (1976) (see Harvey et al. 2003 and Jondeau and Rockinger 2006 for recent developments), it is known that this link between the third derivative of the utility function and the skewness can be simply illustrated by taking a Taylor expansion of the expected utility of the final wealth (w_f) of an investor around his expected wealth ($\overline{w}$) as follows:

$$\begin{aligned} u(w_f) = u(\overline{w}) + u'(\overline{w})(w_f - \overline{w}) + \frac{u''(\overline{w})}{2}(w_f - \overline{w})^2 \\ + \frac{u'''(\overline{w})}{6}(w_f - \overline{w})^3 + \cdots. \end{aligned}$$

This implies

$$\begin{aligned} \mathrm{E}[u(w_f)] = \mathrm{E}[u(\overline{w})] + u'(\overline{w})\,\mathrm{E}[(w_f - \overline{w})] \\ + \frac{u''(\overline{w})}{2}\,\mathrm{E}[(w_f - \overline{w})^2] \\ + \frac{u'''(\overline{w})}{6}\,\mathrm{E}[(w_f - \overline{w})^3] + \cdots, \end{aligned}$$

which finally leads to the expression

$$\mathrm{E}[u(w_f)] = u(\overline{w}) + \frac{u''(\overline{w})}{2}\mathrm{Var}[w_f] + \frac{u'''(\overline{w})}{6}\,\mathrm{Sk}[w_f] + \cdots.$$

Clearly, $u''(\cdot)$ and $u'''(\cdot)$ are respectively related to variance and skewness: while a negative second derivative of the utility function implies variance aversion, a positive third derivative of the utility function entails a preference for positive skewness.[14] Along this line, we define an MVS utility function and a corresponding indirect utility function:

Definition 4.1. The function $U_{(\mu,\rho,\kappa)}\colon \Im \to \mathbb{R}$ defined as

$$U_{(\mu,\rho,\kappa)}(x) = \mu\,\mathrm{E}[R(x)] - \rho\mathrm{Var}[R(x)] + \kappa\,\mathrm{Sk}[R(x)]$$

is called *the MVS utility function*. The function $U^*\colon \mathbb{R}^3_+ \to \mathbb{R}$ defined as

$$U^*(\mu,\rho,\kappa) = \max\left\{U_{(\mu,\rho,\kappa)}(x);\ \sum_{i=1\ldots n} x_i = 1,\ x \geq 0\right\}$$

is called the *indirect MVS utility function*.

This nonlinear optimization program can be rewritten as follows:

$$\begin{aligned} \max\ & \mathrm{E}[R(x)] - \varphi\mathrm{Var}[R(x)] + \Psi\,\mathrm{Sk}[R(x)] \\ \text{s.t.}\ & \sum_{i=1\ldots n} x_i = 1, \quad x \geq 0, \end{aligned} \tag{11}$$

where $\varphi = \rho/\mu \geq 0$ and $\Psi = \kappa/\rho \geq 0$, where the latter ratio represents the degree of absolute prudence. This utility function satisfies positive marginal utility of expected return and skewness and negative marginal utility of risk. Therefore, the maximum value function for the decision maker is simply determined for a given set of parameters $(\mu,\rho,\kappa) > 0$ representing his/her absolute risk aversion and absolute prudence. Knowledge of these parameters allows selecting a

[14] This positive preference direction for the third moment is widely accepted: see, e.g., Kane (1982) or Scott and Horvath (1980). The determination of the preference direction of the fourth moment in relation to the first three moments has been treated in Scott and Horvath (1980). However, Brockett and Kahane (1992) cast serious doubt on the leap from derivatives of utility functions to preferences for moments of arbitrary distributions. Jondeau and Rockinger (2006) describe some recent literature investigating under which conditions adding higher moments improves or deteriorates the approximation.

unique efficient portfolio among those on the weakly efficient frontier maximizing the decision maker's direct MVS utility function.

Lai (1991) and Konno and Suzuki (1995) mention the possibility of directly optimizing this third-order approximation of expected utility, but decline it as impractical given the difficulty of specifying the necessary parameters.[15] This same dual approach is effectively pursued by Jondeau and Rockinger (2006) and Harvey et al. (2003), among others. Because the objective function is nonconcave, it is impossible to guarantee global optimality in the dual approach. By its very nature, one can only verify whether conditions of local optimality are satisfied.

4.3. A Duality Result Between the Hyper-Shortage Function and the Mean-Variance-Skewness Utility Function

Since the representation set $\mathscr{DR}$ is incompatible with a dual representation because of its nonconvexity, we can define the *convex representation set* as follows:

$$\mathscr{CR} = \big\{(E, V, S) \in \mathbb{R}^3;\ \forall (\mu, \rho, \kappa) \in \mathbb{R}^3_+,\ U^*(\mu, \rho, \kappa) \geq \mu E - \rho V + \kappa S\big\}. \quad (12)$$

Basically, $\mathscr{DR}$ is convexified by imposing tangent iso-utility surfaces compatible with the set of admissible MVS portfolios. Now we are in a position to define another shortage function corresponding to $\mathscr{CR}$ and state its properties.[16]

Definition 4.2. Let $g = (g_E, -g_V, g_S) \in \mathbb{R}_+ \times (-\mathbb{R}_+) \times \mathbb{R}$. The function $\bar{S}_g\colon \Im \to \mathbb{R}_+$ defined as $\bar{S}_g(x) = \sup\{\delta;\ \Phi(x) + \delta g \in \mathscr{CR}\}$ is the hyper-shortage function for portfolio x in the direction of vector g.

Proposition 4.3. *$\bar{S}_g$ satisfies the following properties*:

(a) *If $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$, then $\bar{S}_g(x) = 0 \Leftrightarrow x \in \Theta^M(\Im)$ (weak efficiency).*

(b) *$\bar{S}_g$ is MVS-weakly monotonic, i.e.,*

$$\big(\mathrm{E}[R(x')], -\mathrm{Var}[R(x')], \mathrm{Sk}[R(x')]\big) \leq \big(\mathrm{E}[R(x)], -\mathrm{Var}[R(x)], \mathrm{Sk}[R(x)]\big)$$

implies that $0 \leq \bar{S}_g(x) \leq \bar{S}_g(x')$.

(c) *$\bar{S}_g$ is continuous.*

This hyper-shortage function defined on $\mathscr{CR}$ shares almost all the properties of S_g mentioned in Proposition 3.2. Its proof is similar and therefore discarded.

To grasp duality in our framework, it is useful to distinguish between overall, allocative, convexity, and portfolio efficiency when evaluating the scope for improvements in portfolio management.[17] The following definition clearly distinguishes between these concepts. For all $(E, V, S) \in \mathscr{DR}$ and $(\mu, \rho, \kappa) \in \mathbb{R}^3$, we denote

$$(\mu, -\rho, \kappa) \cdot (E, V, S) = \mu E - \rho V + \kappa S.$$

Definition 4.4. The overall efficiency (OE) index is defined as the quantity

$$\mathrm{OE}(x; \mu, \rho, \kappa) = \sup\big\{\delta;\ (\mu, -\rho, \kappa) \cdot (\Phi(x) + \delta g) \leq U^*(\mu, \rho, \kappa)\big\}.$$

The allocative efficiency (AE) index is defined as the quantity $\mathrm{AE}(x; \mu, \rho, \kappa) = \mathrm{OE}(x; \mu, \rho, \kappa) - \bar{S}_g(x)$. The convexity efficiency (CE) index is defined as the quantity $\mathrm{CE}(x) = \bar{S}_g(x) - S_g(x)$. The portfolio efficiency (PE) index is defined as the quantity $\mathrm{PE}(x) = S_g(x)$.

Portfolio efficiency only guarantees reaching a point on the nonconvex primal portfolio frontier, not necessarily a point on the frontier maximizing the investor's indirect MVS utility function. In this sense, it is similar to the notion of technical efficiency in production theory. *Convexity efficiency* measures the difference between the shortage functions computed on both the convex representation set $\mathscr{CR}$ and the initial nonconvex representation set $\mathscr{DR}$. *Allocative efficiency* measures the portfolio adjustment along the convexified portfolio frontier to achieve the maximum of the indirect MVS utility function. This may imply reshuffling an eventual portfolio efficient and convexity efficient portfolio in function of relative prices (i.e., the parameters of the MVS utility function). Finally, *overall efficiency* ensures that all these ideals are achieved simultaneously. In fact, OE is simply the ratio of (i) the difference between indirect MVS utility (Definition 4.1) and the value of the direct MVS utility function for the observation evaluated, and (ii) the normalized value of the direction vector g for given parameters (μ, ρ, κ):

$$\mathrm{OE}(x; \mu, \rho, \kappa) = \frac{U^*(\mu, \rho, \kappa) - U_{(\mu, \rho, \kappa)}(x)}{\mu g_E + \rho g_V + \kappa g_S}. \quad (13)$$

Obviously, the following additive decomposition identity holds:

$$\mathrm{OE}(x; \mu, \rho, \kappa) = \mathrm{AE}(x; \mu, \rho, \kappa) + \mathrm{CE}(x) + \mathrm{PE}(x). \quad (14)$$

Luenberger (1995) established duality between the expenditure function and the shortage function. Similarly, the following result proves that the hyper-shortage function can be computed over the dual of the MVS space. The support function of the representation set $\mathscr{CR}$ is the indirect MVS utility function U^*.

Proposition 4.5. *Let $\bar{S}_g$ be the hyper-shortage function defined on $\Im$. $\bar{S}_g$ has the following properties*:

[15] Konno and Suzuki (1995) also develop a piecewise linear approximation for this direct utility maximization approach.

[16] This development is partly inspired by the way Luenberger (1992) defines the hyper-benefit function in a similar nonconvex setting.

[17] This framework from production theory was transposed to portfolio analysis in Briec et al. (2004).

(1) *For all* $x \in \Im$,

$$\bar{S}_g(x) = \inf_{(\mu,\rho,\kappa)\geq 0}\{U^*(\mu,\rho,\kappa) - U_{(\mu,\rho,\kappa)}(x);\ \mu g_E + \rho g_V + \kappa g_S = 1\}.$$

(2) *For all* $(\mu,\rho,\kappa) \in \mathbb{R}^3_+$,

$$U^*(\mu,\rho,\kappa) = \sup_{x\in\Im}\{U_{(\mu,\rho,\kappa)}(x) - \bar{S}_g(x)\}.$$

Proof. The proof follows from Luenberger (1995). □

Remark 4.6. The special cases of S_g in Remark 3.4 lead to three equivalent special cases for $\bar{S}_g$: (a) return maximization ($g = (g_E, 0, 0)$), (b) risk minimization ($g = (0, -g_V, 0)$), and (c) skewness maximization ($g = (0, 0, g_S)$). These three approaches imply particular versions of the above duality result with extreme investor preferences in terms of absolute risk aversion and absolute prudence.

It is clear that these special types of projections on the MVS frontier are only compatible with rather extreme investor preferences. Note first that Proposition 4.5 is established for all nonnegative triplets (g_E, g_V, g_S). Hence, because the hyper-shortage function is derived from the convex set $\mathscr{CR}$, the above duality result holds true in these special cases. Returning now to some of the examples commented on after Remark 3.4, for instance, it is clear that Joro and Na (2006) focus on variance reduction and Konno and Suzuki (1995) only on skewness augmentation. In our view, these alternative methods may sometimes lead to points of the unknown MVS frontier that are probably unattractive from the viewpoint of general investor preferences (e.g., when a projection is made on vertical or horizontal parts of the nonconvex portfolio frontier). This fact is to some extent masked by these methods because the link to a dual (utility-based) approach is missing. By contrast, the standard shortage function adapted here, by simultaneously looking for improvements in return, risk, and skew, is compatible with what are widely supposed to be more general investor preferences.

4.4. Shadow Prices: Conditions for Their Validity

Next, we devote some attention to study the properties of the shortage function that presume differentiability at the point where the function is evaluated. To this end, the *adjusted risk aversion and prudence function* is introduced:

$$(\mu,\rho,\kappa)(x) = \arg\min\{U^*(\mu,\rho,\kappa) - U_{(\mu,\rho,\kappa)}(x);\ \mu g_E + \rho g_V + \kappa g_S = 1, (\mu,\rho,\kappa) \geq 0\}, \quad (15)$$

that implicitly characterizes both the agent's risk aversion and prudence.[18] Another possible name is the shadow indirect MVS utility function because it adopts a reverse approach by determining the parameters (μ,ρ,κ) and their implied shadow risk aversion and shadow absolute prudence that render the current portfolio optimal for the investor. Note that for these parameters (μ,ρ,κ): OE = PE because AE = 0 and CE = 0 by definition.

The fact that, in principle, absolute risk aversion and absolute prudence can be revealed using this adjusted risk aversion and prudence function expands our possibilities to directly optimize the third-order approximation of expected utility indicated in Definition 4.1 above based on "realistic" parameter values. Therefore, we are slightly more optimistic than, e.g., Lai (1991) about the potential of specifying the necessary parameters.

Proposition 4.7. *Let $\bar{S}_g$ be the hyper-shortage function defined on $\Im$. At the points where $\bar{S}_g$ is differentiable, it has the following properties*:

(1)
$$\frac{\partial \bar{S}_g(x)}{\partial x} = \frac{\partial U_{(\mu,\rho,\kappa)(x)}(x)}{\partial x} = \mu(x)M - 2\rho(x)\Omega x + 3\kappa(x)\Lambda(x\otimes x).$$

(2)
We have

(i) $$\left.\frac{\partial \bar{S}_g(x)}{\partial\, \mathrm{E}[R(x)]}\right|_{\mathrm{Var}[R(x)]=Ct,\, \mathrm{Sk}[R(x)]=Ct} = \mu(x),$$

(ii) $$\left.\frac{\partial \bar{S}_g(x)}{\partial \mathrm{Var}[R(x)]}\right|_{\mathrm{E}[R(x)]=Ct,\, \mathrm{Sk}[R(x)]=Ct} = -\rho(x), \quad \text{and}$$

(iii) $$\left.\frac{\partial \bar{S}_g(x)}{\partial\, \mathrm{Sk}[R(x)]}\right|_{\mathrm{E}[R(x)]=Ct,\, \mathrm{Var}[R(x)]=Ct} = \kappa(x),$$

where M denotes the vector of expected asset returns, Ω is the co-variance matrix, and Λ is the modified co-skewness matrix.

Proof. See the online appendix.

In result (1), it is shown that changes of the hyper-shortage function with respect to x are identical to the variation of the indirect utility function, computed with respect to the adjusted risk aversion and prudence function. Furthermore, this same variation can be linked to the return of each asset, the co-variance and co-skewness matrices. Finally, result (2) shows that the hyper-shortage function increases when the expected return or the skewness increases, or when the variance decreases.

Turning again to the computational aspects, the only requirement to obtain the decomposition from Definition 4.4 is to compute the additional cubic program from Definition 4.1. Then, applying expression (13) and Definition 4.4 itself, the components OE on the one hand and the sum of both components

[18] Luenberger (1995) defines an adjusted price function in consumer theory. Due to its similarity, we label it the *adjusted risk aversion and prudence function*.

AE and CE on the other hand follow from taking the difference between OE and PE. However, because we know of no practical way to compute the hyper-shortage function $\bar{S}_g$, we cannot sharply distinguish between AE and CE.

In contrast to the shortage function, one cannot guarantee global optimality for OE in the dual approach because of the nonconcave nature of the objective function. However, despite the fact that conditions of local optimality do not guarantee global optimality, there is a simple way to detect certain deviations of global optimality for the indirect MVS utility function.

Remark 4.8. In some circumstances, one can infer the nature of the dual optimal solution:

(a) When $\mathrm{PE} = 0$ and overall efficiency (hence also allocative efficiency) turns out to be negative, then the current optimal solution for the indirect utility function ($U^*(\mu, \rho, \kappa)$) is not a global optimum.

(b) When $\mathrm{PE} = 0$, then one cannot infer global optimality for the same indirect utility function.

This finding may well imply that it is better to develop portfolio optimization approaches using a primal rather than a dual framework. However, the development of global optimization algorithms may well soon solve this problem from a computational point of view.

Although the distinction between AE and CE cannot be made, there is a way to determine whether CE is equal to or larger than zero. It suffices to compute PE and to insert its shadow prices as parameters in the objective function when computing OE. If both these components yield identical optimal portfolio weights, then $\mathrm{CE} = 0$ (hence, $\bar{S}_g = S_g$). Otherwise, $\mathrm{CE} > 0$, although its precise magnitude remains unknown. This is expressed more exactly in the following proposition.

Proposition 4.9. *Let $k \in \{1, \dots, m\}$ such that program* (P_2) *has a regular optimal solution. Let $\lambda_E \geq 0$, $\lambda_V \geq 0$, and $\lambda_S \geq 0$ be, respectively, the Kuhn-Tucker multipliers of the first three constraints in program* (P_2). *If S_g is differentiable at point $y^k \in \Im$, and if*

$$y^k \in \arg\max\{U_{(\lambda_E, \lambda_V, \lambda_S)}(x);\ x \in \Im\},$$

then $\mathrm{CE}(y^k) = 0$.

Proof. See the online appendix.

It turns out that Proposition 4.9 is especially of great practical significance when $\mathrm{CE} = 0$ because in that case the shadow prices from PE are identical to the ones of $\bar{S}_g$ (because $S_g = \bar{S}_g$). Thus, the adjusted risk aversion and prudence function (14) can be derived from the Kuhn-Tucker multipliers in program (P_2) when $\mathrm{CE} = 0$ for a specific portfolio y^k under evaluation, as shown in the next proposition.

Proposition 4.10. *Let $k \in \{1, \dots, m\}$ such that program* (P_2) *has a regular optimal solution. Let $\lambda_E \geq 0$, $\lambda_V \geq 0$, and $\lambda_S \geq 0$ be, respectively, the Kuhn-Tucker multipliers of the first three constraints in program* (P_2). *If S_g is differentiable at point $y^k \in \Im$, and if there exists a neighborhood $V(y^k, \epsilon)$ such that $\mathrm{CE}(y) = 0$ for all $y \in V(y^k, \epsilon)$, then this yields*

(1)
$$\left.\frac{\partial S_g(y)}{\partial\, \mathrm{E}[R(y)]}\right|_{\substack{y = y^k \\ \mathrm{Var}[R(y)] = \mathrm{Var}[R(y^k)] \\ \mathrm{Sk}[R(y)] = \mathrm{Sk}[R(y^k)]}} = \lambda_E,$$

$$\left.\frac{\partial S_g(y)}{\partial \mathrm{Var}[R(y)]}\right|_{\substack{y = y^k \\ \mathrm{E}[R(y)] = \mathrm{E}[R(y^k)] \\ \mathrm{Sk}[R(y)] = S[R(y^k)]}} = -\lambda_V, \quad \textit{and}$$

$$\left.\frac{\partial S_g(y)}{\partial\, \mathrm{Sk}[R(y)]}\right|_{\substack{y = y^k \\ \mathrm{E}[R(y)] = E[R(y^k)] \\ \mathrm{Var}[R(y)] = \mathrm{Var}[R(y^k)]}} = \lambda_S.$$

(2) *The adjusted risk aversion and prudence function is identical to the Kuhn-Tucker multipliers*:

$$(\mu, \rho, \kappa)(y^k) = (\lambda_E, \lambda_V, \lambda_S).$$

Proof. See the online appendix.

Note that this last result only holds true when $\mathrm{CE} = 0$.

To conclude, the introduction of the hyper-shortage function only serves to establish the above duality result and to obtain an economic interpretation for the initial shortage function. The fact that the hyper-shortage function cannot be computed creates no practical difficulties because it is in general not meaningful to obtain estimates of shadow risk aversion and prudence for all observations based on the hyper-shortage function. Shadow prices are only meaningful if the convexity efficiency is zero because only then the initial shortage function and the hyper-shortage function coincide. If both functions coincide, then the shadow prices coincide too (see Proposition 4.10). Proposition 4.9 establishes a simple way to verify whether $\mathrm{CE} = 0$ or not, and thus whether the shadow prices of the initial shortage function have an economic meaning.

In general, it would of course be desirable to have a way of computing the hyper-shortage function $\bar{S}_g$. First, this would allow to separate AE and CE sharply instead of only being able to determine whether $\mathrm{CE} = 0$. Second, $\bar{S}_g$ could also be instrumental in the computation of the indirect MVS utility function. Indeed, starting from a projection of an initial (eventually inefficient) portfolio using $\bar{S}_g$ onto the boundary of $\mathscr{CR}$, computing OE (Definition 4.4) with current optimization tools would guarantee a global optimum.

5. Empirical Illustration: Assets Composing the French CAC40 Index

Just as an empirical illustration, we compute the decomposition of overall efficiency for a small sample of 35 assets being part of the French CAC40 index between February 1997 and October 1999.[19] This sample contains 567 daily return observations in common for all assets on which the first three centered moments have been computed. As stated before, our analysis can be applied to both assets and funds when keeping the proper interpretation in mind. When evaluating assets, each of the assets in turn is projected onto the MVS frontier and furthermore evaluated with respect to the optimal point on the same frontier given certain parameters of the indirect MVS utility function. This yields an optimal portfolio starting from a given asset with specific characteristics. This perspective may seem unusual, but it should be kept in mind that our approach does not try to trace the whole frontier, but only evaluates existing assets relative to this same frontier. When evaluating funds, each fund is projected onto the frontier and evaluated against an optimal point on the frontier in an effort to define a fund of funds. This adheres to a more traditional interpretation.

The calculation of the cubic program (P_2) yields PE. Then, solving the cubic program (11) with parameters $\mu = 1$, $\rho = 1.5$, and $\kappa = 1.5$ determines the maximum of the indirect MVS utility function in Definition 4.1. These parameters of the MVS utility function fix absolute risk aversion ($\varphi = 1.5$) and absolute prudence ($\Psi = 1$) around conventional values. Finally, applying the decomposition in Definition 4.4 and using (13) leads to the decomposition results in Table 1. Note that our AE component also includes CE: that is, no effort was done to determine whether CE is larger than or equal to zero. To save space, portfolio weights and slack variables are not reported. These results are contrasted with the MV results using basically (P_2) without the skewness constraint and quadratic program (10) (see Briec et al. 2004 for all details).

A technical remark on the choice of a direction vector when computing (P_2) needs to be added. The direction vector retained is the return, variance, and skewness of the evaluated asset itself. This turns the shortage function into a proportional shortage function: return and skewness are proportionally increased, while variance is proportionally reduced. In particular, we assume that $g_E = |\mathrm{E}[R(x)]|$, $g_V = \mathrm{Var}[R(x)]$, and $g_S = |\mathrm{Sk}[R(x)]|$. Taking absolute values of return and skewness is needed because one cannot

[19] Changes in the CAC40 index over this period explain our focus on these 35 out of the total of 40 assets.

Table 1 **Mean-Variance-Skewness vs. Mean-Variance Benchmarking**

	Assets from CAC40	Mean-variance-skewness			Mean-variance		
		OE	AE*	PE	OE	AE	PE
1.	Accor	1.192	0.366	0.826	1.082	0.256	0.826
2.	AGF	7.174	6.587	0.587	9.634	9.047	0.587
3.	AirLiquide	2.426	1.596	0.830	2.049	1.219	0.830
4.	Alcatel	1.385	0.469	0.916	1.266	0.350	0.916
5.	Aventis	1.640	1.640	0.000	1.468	0.598	0.870
6.	AXA	1.206	0.607	0.600	1.084	0.327	0.757
7.	BNP	0.989	0.989	0.000	0.935	0.094	0.841
8.	Bouygues	0.334	0.000	0.333	0.341	0.007	0.334
9.	Capgemini	1.137	0.248	0.889	1.053	0.164	0.889
10.	Carrefour	1.121	1.121	0.000	1.019	0.261	0.758
11.	Casino	1.331	0.612	0.719	1.182	0.462	0.720
12.	CreditLyonnais	0.602	0.602	0.000	0.566	0.046	0.520
13.	Danone	1.964	1.199	0.766	1.700	0.935	0.765
14.	Dassault	0.996	0.159	0.837	0.933	0.096	0.837
15.	Dexia	2.726	1.957	0.770	2.292	1.522	0.770
16.	Lafarge	1.421	0.827	0.594	1.270	0.428	0.843
17.	Lagardere	1.674	0.787	0.887	1.469	0.582	0.887
18.	L'Oreal	1.731	1.034	0.697	1.514	0.648	0.865
19.	LVMH	1.195	1.195	0.000	1.092	0.266	0.826
20.	Michelin	2.851	2.207	0.644	3.487	2.602	0.885
21.	Peugeot	1.247	0.412	0.835	1.127	0.292	0.835
22.	PPR	0.896	0.704	0.192	0.836	0.103	0.733
23.	Renault	0.897	0.897	0.000	0.860	0.042	0.819
24.	Gobain	1.808	0.963	0.844	1.571	0.727	0.843
25.	Sanofi	1.570	1.570	0.000	1.396	0.534	0.861
26.	Schneider	2.742	1.849	0.893	2.228	1.335	0.893
27.	SocGenerale	1.119	0.271	0.848	1.029	0.181	0.848
28.	Sodhexo	1.819	0.971	0.847	1.592	0.745	0.847
29.	ST Micro	0.102	0.102	0.000	0.167	0.167	0.000
30.	Suez	2.513	2.383	0.129	2.140	1.392	0.748
31.	TF1	0.330	0.330	0.000	0.341	0.009	0.332
32.	Thales	2.269	1.366	0.903	1.898	0.995	0.903
33.	Total	1.318	0.492	0.827	1.183	0.356	0.827
34.	Vinci	0.922	0.580	0.342	0.855	0.115	0.741
35.	VivendiUniversal	1.573	1.573	0.000	1.389	0.648	0.740
Mean		1.606	1.105	0.502	1.544	0.787	0.757
Std. dev.		1.183	1.133	0.374	1.544	1.538	0.191
Max		7.174	6.587	0.916	9.634	9.047	0.916

*Includes CE.

preclude negative values. In practice, this amounts to taking a positive (negative) δ in (P_2) for positive (negative) values of return and skewness.

To develop some intuition with the above theoretical developments, we first interpret the decomposition results for a few single assets. First, we focus on the asset "Vinci" and show how the above procedures can be applied in practice. Then, as an illustration of the fact that sometimes the differences between the MVS and MV results are wide, we discuss the asset "Credit Lyonnais." Thereafter, we make some comments on the sample results on the average.

Example 5.1. For the single asset "Vinci," its initially observed mean return is 0.0013, its risk is 0.00056, and its skewness is 2.9258E−06. These observed values are entered on the lhs of program (P_2), and the model is solved. Holding all wealth in this asset and projecting using its direction vector leads to a portfolio that is doing 92% better in terms of OE compared to this asset. That is, by applying the optimal portfolio weights one can simultaneously improve return and skewness and reduce risk of this same asset by 92%. The decomposition indicates that 34% of this poor performance is due to PE (i.e., operating below the nonconvex portfolio frontier), while the remaining 58% of the gap is due to AE (i.e., choosing a wrong mix of return, skewness, and risk given postulated risk aversion and prudence parameters). At the PE optimum, its return has increased to 0.0022, its risk has been reduced to 0.00037, and its skewness has risen to 3.9253E−06. The optimal weights for this solution are $x_3 = 0.056$, $x_8 = 0.087$, $x_9 = 0.031$, $x_{26} = 0.096$, $x_{29} = 0.480$, $x_{32} = 0.165$, and $x_{33} = 0.085$. By contrast, in the traditional MV model we obtain a PE optimum with a higher return of 0.0023 and a lower risk of 0.00014, but its skewness has now actually decreased to 2.88173E−07 compared to its initial skewness. The optimal weights are now $x_8 = 0.174$, $x_{10} = 0.035$, $x_{11} = 0.023$, $x_{12} = 0.250$, $x_{14} = 0.012$, $x_{22} = 0.013$, $x_{23} = 0.023$, $x_{29} = 0.214$, $x_{31} = 0.190$, and $x_{34} = 0.065$. Note that the MVS model implies an average nonzero weight of 0.143 and a maximum weight of 0.480, while the MV model leads to an average nonzero weight of 0.1 and a maximum weight of 0.250. Thus, the MVS model leads to less diversification compared to the MV model in an effort to win in terms of skewness. By contrast, the MV model offers better results in terms of return and risk, but at the cost of ignoring the skewness dimension altogether.

Example 5.2. "Credit Lyonnais" is deemed very portfolio inefficient in MV space, while it is spanning the MVS frontier (PE = 0). Starting off from an observed return of 0.0017, a risk of 0.00041, and a skewness of 9.57366E−06, the MVS model claims that these result cannot be improved on, while the MV model yields a PE improved optimum return of 0.0026 and a reduced risk of 0.00020, but at the cost of reducing the skewness to only 5.84529E−07. Thus, the performance improvement suggested by the MV model turns out to be completely illusory: in fact, no improvement can be made once the skewness dimension is taken into account.

The average performance of the individual assets is poor. In MVS space, they could improve their OE performance by about 160% (compared to 154% in MV space). The decomposition results indicate that the majority of these inefficiencies can be attributed to AE (compared to PE in MV space). Average portfolio inefficiency is only about 50% (compared to about 76% in MV space). When looking at individual assets, no single asset perfectly corresponds to the investors' preferences in that the minimum OE is 10% ("ST Micro"). However, in total 10 assets are portfolio efficient and span the MVS frontier (compared to only one asset in the MV space). Obviously, as stated above, PE in MVS space is always smaller or equal to PE in MV space because of the additional constraint. This explains, for instance, why the first three assets have identical PE in both spaces.

Table 2 Optimal Portfolio Composition

Portfolio models	Number of nonzero weights	Average weight*	Std. dev. weight
Mean-variance-skewness	6.12	0.163	0.420
MVS: Maximum return	1.42	0.705	0.331
MVS: Minimum risk	8.78	0.114	0.379
MVS: Maximum skewness	1.28	0.779	0.273
Mean-variance	12.45	0.080	0.168

Note. MVS = Mean-variance-skewness.
*Geometric mean.

Table 2 reports in a condensed form the distribution of the optimal portfolio weights. In particular, we report the number of nonzero weights as well as the mean and standard deviation of these portfolio weights. Furthermore, the portfolio weights corresponding to the following approaches are contrasted: the shortage function in full MVS space, as well as its three special cases, the (i) maximum return, (ii) minimum risk, and (iii) maximum skewness models.

Comparing the MVS and the MV results first, one observes that the latter implies a higher diversification with on average lower weights and less dispersion among weights. The minimum risk model resembles the MV approach in that, on average, it has 8.78 nonzero portfolio weights. These weights are somewhat higher than the MV weights, but lower than the optimal MVS portfolio weights. The maximum return and maximum skewness models turn out to generate rather extreme solutions by concentrating wealth in less than two assets with extremely high-average weights as a consequence. This casts some doubts on the approaches in the literature advancing these modeling strategies.

An effort was done to determine whether CE is larger than or equal to zero. It turns out that for eight out of 35 observations, CE = 0. However, among these eight observations, several observations contain some slack(s) in one of the three dimensions. Therefore, it is hard (if not impossible) to obtain reliable information on the implied shadow risk aversion and prudence. If one could obtain reliable estimates for enough observations in the sample, then, in principle, the confrontation between postulated risk aversion and prudence parameters and the shadow risk aversion and shadow prudence would allow inferring whether actual portfolio management strategies conform to certain ideal pre-specified risk aversion and prudence profiles.

As a final remark, it is worthwhile pointing out that decomposition results depend on specific risk aversion and prudence parameters. But, if one is reluctant to specify these parameters, then nothing prevents one from simply computing PE while ignoring OE and AE. The only inconvenience may be that it may be difficult for an investor to have a clear idea about the position of certain portfolio efficient points in a three-dimensional MVS space. The specification of an indirect MVS utility function has the advantage of picking an optimal point without the need to consider the exact geometry of the three-dimensional frontier.

6. Conclusions

This paper has introduced a general method for benchmarking portfolios in the nonconvex MVS space using the shortage function framework (Luenberger 1995). Portfolio efficiency is evaluated by looking for risk contraction on the one hand, and mean return and skewness augmentation on the other hand. This shortage function is linked to an indirect MVS utility function. Exploiting this duality allows to differentiate between portfolio efficiency, allocative efficiency, and a convexity efficiency component. The latter component is related to the difference between the primal, nonconvex approach and the dual, convex approach. A brief empirical application has served to illustrate the computational tractability of the approach.

The proposed framework approximates the true frontier by a nonparametric frontier using an efficiency measure that is perfectly suitable for performance gauging and that guarantees global optimality. In addition, this shortage function can specialize to any of the existing approaches focusing on return maximization, skewness maximization, or risk minimization. Further virtues are that interesting dual interpretations are available without imposing any simplifying hypotheses. Unfortunately, no global optimal solution can be guaranteed for the indirect MVS utility function. These findings indicate that future research should probably focus on developing portfolio optimization methods using a primal rather than a dual approach.

One could first of all hope for some further improvements in the proposed framework. For instance, it would be good to have a computational procedure to obtain the hyper-shortage function because this would enable identifying the convexity efficiency component. Furthermore, the recent development of proper statistical inference for nonparametric frontier models in a production context could probably be transposed in an investment context (see Simar and Wilson 2000).

But more drastic extensions are possible. Because the shortage function is a distance function capable of representing multidimensional choice sets, one obvious theoretical extension is to treat the general, higher-order moment portfolio problem corresponding to a general, higher-order Taylor expansion of the expected utility function. This would, for instance, allow integrating the full kurtosis–co-kurtosis matrix into the current MVS portfolio-gauging framework. This would allow to improve on the recent efforts of, e.g., Athayde and Flôres (2003) who come up with a mean-skewness-kurtosis model, but they ignore the variance dimension. Because the transition from the traditional MV to the MVS space necessitated dealing with nonconvexities, one could hope these further generalizations would not be hindered by too many computational problems.

An e-companion to this paper is available as part of the online version that can be found at http://mansci.pubs.informs.org/.

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References

Arditti, F., H. Levy. 1975. Portfolio efficiency analysis in three moments: The multiperiod case. *J. Finance* **30** 797–809.

Athayde, G., R. Flôres. 2003. Incorporating skewness and kurtosis in portfolio optimization: A multidimensional efficient set. S. Satchell, A. Scowcroft, eds. *Advances in Portfolio Construction and Implementation*. Butterworth-Heinemann, Oxford, UK, 243–257.

Athayde, G., R. Flôres. 2004. Finding a maximum skewness portfolio: A general solution to three-moments portfolio choice. *J. Econom. Dynam. Control* **28** 1335–1352.

Briec, W., K. Kerstens, J.-B. Lesourd. 2004. Single period Markowitz portfolio selection, performance gauging and duality: A variation on the Luenberger shortage function. *J. Optim. Theory Appl.* **120** 1–27.

Brockett, P. L., Y. Kahane. 1992. Risk, return, skewness and preference. *Management Sci.* **38** 851–866.

Chunhachinda, P., K. Dandapani, S. Hamid, A. J. Prakash. 1997. Portfolio selection and skewness: Evidence from international stock markets. *J. Banking Finance* **21** 143–167.

Deaton, A. 1979. The distance function in consumer behaviour with applications to index numbers and optimal taxation. *Rev. Econom. Stud.* **46** 391–405.

Farrar, D. E. 1962. *The Investment Decision Under Uncertainty*. Prentice-Hall, Englewood Cliffs, NJ.

Fiacco, A. V., G. P. McGormick. 1968. *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*. John Wiley, New York.

Harvey, C. R., J. C. Liechty, M. W. Liechty, P. Müller. 2003. Portfolio selection with higher moments. Mimeo, Drexel University, Philadelphia, PA.

Jensen, M. 1968. The performance of mutual funds in the period 1945–1964. *J. Finance* **23** 389–416.

Jobst, N. J., M. D. Horniman, C. A. Lucas, G. Mitra. 2001. Computational aspects of alternative portfolio selection models in the presence of discrete asset choice constraints. *Quant. Finance* **1** 489–501.

Jondeau, E., M. Rockinger. 2006. Optimal portfolio allocation under higher moments. *Eur. Financial Management* **12** 29–55.

Joro, T., P. Na. 2006. Portfolio performance evaluation in mean-variance-skewness framework. *Eur. J. Oper. Res.* **175**(1) 516–542.

Kane, A. 1982. Skewness preference and portfolio choice. *J. Financial Quant. Anal.* **17** 15–25.

Kimball, M. S. 1990. Precautionary saving in the small and in the large. *Econometrica* **58** 53–73.

Konno, H., K. Suzuki. 1995. A mean-variance-skewness portfolio optimization model. *J. Oper. Res. Soc. Japan* **38** 173–187.

Kraus, A., R. H. Litzenberger. 1976. Skewness preference and the valuation of risky assets. *J. Finance* **31** 1085–1100.

Lai, T. Y. 1991. Portfolio selection with skewness: A multiple-objective approach. *Rev. Quant. Finance Accounting* **1** 293–305.

Lien, G. 2002. Non-parametric estimation of decision makers' risk aversion. *Agricultural Econom.* **27** 75–83.

Lintner, J. 1965. The valuation of risk assets and the selection of risky investment in stock portfolios and capital budgets. *Rev. Econom. Statist.* **47** 13–37.

Luenberger, D. G. 1984. *Linear and Nonlinear Programming*, 2nd ed. Addison-Wesley, Reading, MA.

Luenberger, D. G. 1992. New optimality principles for economic efficiency and equilibrium. *J. Optim. Theory Appl.* **75** 221–264.

Luenberger, D. G. 1995. *Microeconomic Theory*. McGraw-Hill, New York.

Markowitz, H. 1952. Portfolio selection. *J. Finance* **7** 77–91.

Markowitz, H. 1959. *Portfolio Selection: Efficient Diversification of Investments*. John Wiley, New York.

Markowitz, H. 1991. Foundations of portfolio theory. *J. Finance* **46** 469–478.

Morey, M. R., R. C. Morey. 1999. Mutual fund performance appraisals: A multi-horizon perspective with endogenous benchmarking. *Omega* **27** 241–258.

Philippatos, G. C. 1979. Alternatives to mean-variance for portfolio selection. J. L. Bicksler, ed. *Handbook of Financial Economics*. North-Holland, Amsterdam, The Netherlands, 365–386.

Pogue, G. 1970. An extension of the Markowitz portfolio selection model to include variable transactions' costs, short sales, leverage policies and taxes. *J. Finance* **25** 1005–1027.

Prakash, A., C.-H. Chang, E. Pactwa. 2003. Selecting a portfolio with skewness: Recent evidence from US, European, and Latin America equity markets. *J. Banking Finance* **27** 1375–1390.

Rudd, A., B. Rosenberg. 1979. Realistic portfolio optimization. E. J. Elton, M. J. Gruber, eds. *Portfolio Theory, 25 Years After*. North-Holland, Amsterdam, The Netherlands, 21–46.

Samuelson, P. A. 1967. General proof that diversification pays. *J. Financial Quant. Anal.* **2** 1–13.

Samuelson, P. A. 1970. The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments. *Rev. Econom. Stud.* **37** 537–542.

Scott, R. C., P. A. Horvath. 1980. On the direction of preference for moments of higher order than the variance. *J. Finance* **35** 915–919.

Sharpe, W. 1963. A simplified model for portfolio analysis. *Management Sci.* **9** 277–293.

Sharpe, W. 1964. Capital asset prices: A theory of market equilibrium under condition of risk. *J. Finance* **19** 425–442.

Sharpe, W. 1966. Mutual fund performance. *J. Bus.* **39** 119–138.

Simar, L., P. W. Wilson. 2000. A general methodology for bootstrapping in non-parametric frontier models. *J. Appl. Statist.* **27** 779–802.

Simkowitz, M. A., W. L. Beedles. 1978. Diversification in a three-moment world. *J. Financial Quant. Anal.* **13** 927–941.

Sun, Q., Y. Yan. 2003. Skewness persistence with optimal portfolio selection. *J. Banking Finance* **27** 1111–1121.

Treynor, J. L. 1965. How to rate management of investment funds. *Harvard Bus. Rev.* **43** 63–75.

Wang, S., Y. Xia. 2002. *Portfolio Selection and Asset Pricing*. Springer, Berlin, Germany.

Womersley, R. S., K. Lau. 1996. Portfolio optimisation problems. R. L. May, A. K. Easton, eds. *Computational Techniques and Applications: CTAC95*. World Scientific Publishing, Singapore, 795–802.