JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 120, No. 1, pp. 1–27, January 2004 (©2004)

Single-Period Markowitz Portfolio Selection, Performance Gauging, and Duality: A Variation on the Luenberger Shortage Function[1]

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Communicated by D. G. Luenberger

Abstract. The Markowitz portfolio theory (Ref. 1) has stimulated research into the efficiency of portfolio management. This paper studies existing nonparametric efficiency measurement approaches for single-period portfolio selection from a theoretical perspective and generalizes currently used efficiency measures into the full mean-variance space. We introduce the efficiency improvement possibility function (a variation on the shortage function), study its axiomatic properties in the context of the Markowitz efficient frontier, and establish a link to the indirect mean-variance utility function. This framework allows distinguishing between portfolio efficiency and allocative efficiency; furthermore, it permits retrieving information about the revealed risk aversion of investors. The efficiency improvement possibility function provides a more general framework for gauging the efficiency of portfolio management using nonparametric frontier envelopment methods based on quadratic optimization.

Key Words. Shortage function, efficient frontier, risk aversion, mean-variance portfolios.

1. Introduction

Markowitz (Ref. 1) seminal work on modern portfolio theory introduced the idea of a tradeoff between risk and expected return of a portfolio;

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and defined an efficient frontier concept as a Pareto-optimal subset of portfolios, that is, portfolios whose expected returns may not increase unless their variances increase. In addition to its strongly maintained assumptions on probability distributions and Von Neumann-Morgenstern utility functions, the main problem at the time was the computational cost of solving quadratic programs. Farrar (Ref. 2) was apparently the first to test empirically the full-covariance Markowitz model, while computing costs motivated Sharpe (Ref. 3) to formulate a simplified diagonal model. Later, Sharpe (Ref. 4) and Lintner (Ref. 5) introduced the capital asset pricing model (CAPM), an equilibrium model assuming that all agents have similar expectations about the market. Under these circumstances, it is not necessary to compute the efficient frontier. Historical surveys of these developments are e.g. Constantinides and Malliaris (Ref. 6) and Philippatos (Ref. 7). Tools for gauging portfolio efficiency, such as the Sharpe (Ref. 8) and Treynor (Ref. 9) ratios and the Jensen (Ref. 10) alpha, have been developed mainly with reference to these developments (in particular CAPM). Surveys on measuring the performance of managed portfolios are found in Grinblatt and Titman (Ref. 11) or Shukla and Trzcinka (Ref. 12).

Despite these enhancements, the static Markowitz model remains the more general framework. Our contribution integrates an efficiency measure into this single-period Markowitz model and develops a dual framework for assessing the degree of satisfaction of the investors preferences, starting from the seemingly forgotten ideas advanced by Farrar (Ref. 2). This leads to decomposing the portfolio performance into allocative and portfolio efficiency components. In addition, this duality offers information about the investors risk aversion via the shadow prices associated with the specific efficiency measure. This is an issue of great practical significance that, to the best of our knowledge, is novel. An empirical application is included to illustrate the potentials of the proposed framework.

There are both theoretical and practical motivations guiding these developments. Theoretically, this contribution brings portfolio theory in line with developments in production theory, where distance functions have proven to be useful tools to derive efficiency measures and to develop dual, relations with economic (e.g. profit) support functions [Chambers, Chung, and Färe (Ref. 13)]. From a practical viewpoint, there are the following advantages. First, the integration of efficiency measures responds to the needs for portfolio rating tools. Second, instead of tracing the whole efficient portfolio frontier using a critical line search method, each asset or fund is projected onto the relevant part of the frontier according to a meaningful efficiency measure. This may lead to computational gains, depending on the number of assets or funds to evaluate and the aimed fineness of the portfolio frontier representation. Third, the possibility of measuring portfolio performance using a dual approach permits not only gauging assets or funds using given information about risk aversion, but it reveals also the (shadow) risk aversion minimizing portfolio inefficiency. In these ways, the contribution enriches the empirical toolbox of practitioners.

A variation of the shortage function is introduced, a distance function proposed in production theory by Luenberger (Ref. 14) that is dual to the profit function. This function accomplishes four goals: (i) it gauges portfolio performance by measuring a distance between a portfolio and an optimal portfolio projection on the Markowitz efficient frontier; (ii) it leads to a nonparametric estimation of an inner bound of the true but unknown portfolio frontier; (iii) it judges simultaneously mean-return expansions and risk contractions and thereby generalizes existing approaches; and (iv) it provides a new, dual interpretation of our portfolio efficiency distance. Given the investment context, this efficiency measure is called the efficiency improvement possibility (EIP) function.

To develop point (iv), the paper establishes a link between the EIP function and mean-variance utility functions, thereby offering an integrated framework for assessing portfolio efficiency from the dual standpoint. To each efficient portfolio, there corresponds a particular utility function, whose optimal value is the indirect utility function. This approach provides a dual interpretation of the EIP function through the structure of risk preferences. Technically, this result is derived easily from Luenberger (Ref. 14–15). Along this line, a link is established to some kind of Slutsky matrix, defined as a matrix of derivatives with respect to risk aversion (based on the structure of the mean-variance utility function).

To situate the results more precisely, it is possible to distinguish between several approaches for testing portfolio efficiency. It is common to develop statistical tests based on certain parametric distributional assumptions [e.g. Jobson and Korkie (Ref. 16), Gouriéroux and Jouneau (Ref. 17), Philippatos (Ref. 7)]. However, from the outset [Markowitz (Ref. 1)], there has been attention also to simple nonparametric approaches to test for portfolio efficiency. This work is best contrasted with some recent developments in the nonparametric test tradition which uses economic restrictions [Matzkin (Ref. 18)]. Varian (Ref. 19) develops nonstatistical tests checking whether the observed investments are consistent with the expected utility and the mean-variance models. However, his formulation can infer only whether or not certain data are consistent with the tested hypothesis, but lacks an indication about the degree of goodness of fit between data and models.[5] Sengupta (Ref. 20) is probably the first to link the Varian (Ref. 19) portfolio test

[5]In this respect, it is similar to the early nonparametric test literature on production [Diewert and Parkan (Ref. 22)] and consumption [Varian (Ref. 23)].

approach to the nonparametric efficiency literature by introducing explicitly an efficiency measure.[6] Morey and Morey (Ref. 21) measure investment fund performance focusing on radial potentials for either risk contraction or mean-return expansion. By contrast, the approach in this article looks simultaneously for risk contraction and mean-return augmentation.

Among the advantages of a nonparametric approach to production,[7] consumption and investment, one can mention: (i) it avoids postulating specific functional forms, (ii) it uses revealed preference conditions of some sort that are finite in nature and that are directly tested on a finite number of observations, (iii) it determines inner and outer approximations of choice sets that contain the true but unknown frontier, (iv) these approximations are based on (most frequently piecewise linear) functions that are spanned directly by the observations in the sample, (v) the computational cost is low, often just solving mathematical programming problems [e.g. Matzkin (Ref. 18), Morey and Morey (Ref. 21), Varian (Ref. 19)].

Section 2 of the article lays down the foundations of the analysis. Section 3 introduces the EIP function and studies its axiomatic properties. Section 4 studies the link between the EIP function and the direct and indirect mean-variance utility functions. Section 5 presents mathematical programs to compute the efficiency decomposition. A simple empirical illustration using a small sample of 26 investment funds is provided in Section 6. Conclusions and possible extensions are formulated in Section 7.

2. Efficient Frontier and Portfolio Management

In developing the basic definitions, consider the problem of selecting a portfolio (or fund of funds) from n financial assets (or funds). Assets are characterized by an expected return $E(R_i)$, $i = 1, \ldots, n$, since returns of assets are correlated by a covariance matrix $\Omega_{i,j} = \mathrm{Cov}(R_i, R_j)$, $i, j \in \{1, \ldots, n\}$. A portfolio x is composed by a proportion of each of these n financial assets. Thus, one can define $x = (x_1, \ldots, x_n)$, with $\sum_{i=1,\ldots,n} x_i = 1$. The condition $x_i \geq 0$ is imposed whenever short sales are excluded.

[6]Färe and Grosskopf (Ref. 24) link the literature on regularity tests and the efficiency contributions employing distance functions (or their inverses, efficiency measures) as an explicit (nonstatistical) goodness of the fit indicator.

[7]Aside from the investment context, the estimation of monotone concave boundaries is extensively studied in production. Following Farrell (Ref. 25), nonparametric efficiency methods estimate an inner bound approximation of the true, unknown production frontier using piecewise linear envelopments of the data, instead of traditional parametric, econometric estimation methods that suffer from the risk of specification error.

Decision makers face often additional economic constraints [see, e.g., Pogue (Ref. 26) or Rudd and Rosenberg (Ref. 27)]. For instance, the proportion of each of the n financial assets composing a portfolio can be modified by taking into account transaction costs or by imposing upper limits on any fraction invested. If these constraints are linear functions of the asset weights, then the set of admissible portfolios is defined as

$$\Im = \left\{ x \in R^n;\ \sum_{i=1\dots n} x_i = 1, Ax \leq b, x \geq 0 \right\}, \tag{1}$$

where A is a $m \times n$ matrix and $b \in R^m$. It is assumed throughout the paper that $\Im \neq \emptyset$.

The return of portfolio x is

$$R(x) = \sum_{i=1,\dots,n} x_i R_i.$$

The expected return and its variance can be calculated as follows:

$$E(R(x)) = \sum_{i=1,\dots,n} x_i E(R_i), \tag{2}$$

$$V(R(x)) = \sum_{i,j} x_i x_j \operatorname{Cov}(R_i, R_j). \tag{3}$$

It is useful to define the mean-variance representation of the set $\Im$ of portfolios. From Markowitz (Ref. 1), it is straightforward to give the following definition:

$$\aleph = \{(V(R(x)), E(R(x))); x \in \Im\}. \tag{4}$$

However, such a representation cannot be used for quadratic programming, because the subset $\aleph$ is not convex [see for instance Luenberger (Ref. 28)]. Thus, the above set can be extended by defining a mean-variance (portfolio) representation set through

$$\Re = \{\aleph + (R_+ \times (-R_+))\} \cap R_+^2\,. \tag{5}$$

This set can be rewritten as follows:

$$\Re = \{(V', E') \in R_+^2; \exists x \in \Im, (-V', E') \leq (-V(R(x)), E(R(x)))\}. \tag{6}$$

The addition of the cone is necessary for the definition of a sort of "free disposal hull" of the mean-variance representation of feasible portfolios. Clearly, the above definition is compatible with the definition in Markowitz (Ref. 1). To measure the degree of portfolio efficiency, it is necessary to isolate a subset of this representation set, generally known as the efficient frontier. This subset is defined as follows.

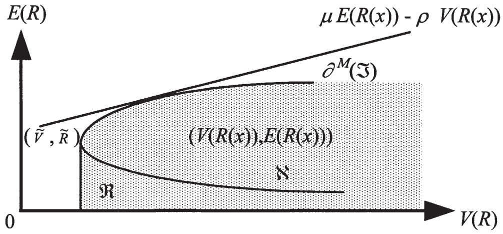

Fig. 1. Finding the optimal portfolio.

Definition 2.1. In the mean-variance space, the weakly efficient frontier is defined as

$$\partial^M(\Im) = \{(V(R(x)), E(R(x)));\ x \in \Im \wedge (-V(R(x)), E(R(x))) < (-V', E') \Rightarrow (V', E') \notin \Re\}.$$

From the above definition, the weakly efficient frontier is the set of all the mean-variance points that are not strictly dominated in the two-dimensional space. It is possible also to define a strongly efficient frontier, but the above formulation simplifies most results in this contribution. Moreover, the geometric representation of the frontier (see Figure 1) is quite similar, except for some rather special cases.

The above definition enables us to define the set of weakly efficient portfolios.

Definition 2.2. The set of the weakly efficient portfolios is defined in the simplex as

$$\Lambda^M(\Im) = \{x \in \Im; (V(R(x)), E(R(x))) \in \partial^M(\Im)\}.$$

Markowitz (Ref. 29) defines an optimization program to determine the portfolio corresponding to a given degree of risk aversion. This portfolio maximizes a mean-variance utility function defined by

$$U_{(\rho,\mu)}(x) = \mu E(R(x)) - \rho V(R(x)), \tag{7}$$

where $\mu \geq 0$ and $\rho \geq 0$. This utility function satisfies positive marginal utility of expected return and negative marginal utility of risk. The quadratic optimization program may be simply written as follows:

$$\max \quad U_{(\rho,\mu)}(x) = \mu E(R(x)) - \rho V(R(x)), \tag{8a}$$

$$\text{s.t.} \quad Ax \leq b, \tag{8b}$$

$$\sum_{i=1,\dots,n} x_i = 1, \qquad x \geq 0. \tag{8c}$$

Traditionally, the ratio $\varphi = \rho/\mu \in [0, +\infty]$ represents the degree of absolute risk aversion.

Setting $\mu = 0$ and $\rho = 1$ eliminates the return information from this quadratic mathematical program and yields the efficient portfolio with minimum risk. Denoting this global minimum-variance portfolio $\tilde{x}$, it can be represented in the two-dimensional mean-variance space as (see Figure 1)

$$(\tilde{V}, \tilde{R}) = (V(R(\tilde{x})), E(R(\tilde{x}))).$$

When shorting is allowed or there is a riskless asset with zero variance and nonzero positive return, then from the two-fund theorem and the one-fund theorem, the efficient frontier is determined by simple analytical solutions [e.g. Elton, Gruber, and Padberg (Ref. 30), or Luenberger (Ref. 28)]. Though the computational burden of the more general quadratic programming approach remains substantial, when building realistic portfolio models it is hard to avoid. The approach developed in Section 3 adheres to this quadratic programming tradition to maintain generality. To extend the well-known Markowitz approach, Section 3 introduces the EIP function of a portfolio as an indicator of its performance. This EIP function is similar to the shortage function [see Luenberger (Ref. 14)].

3. Efficiency Improvement Possibility Function and the Frontier of Efficient Portfolios

Intuitively stated, the shortage function in production theory measures the distance between some point of the production set and the Pareto frontier. Before introducing this function formally in a portfolio context, it is of interest to focus on the basic properties of the subset $\Re$ on which the shortage function is defined below.

Proposition 3.1. The subset $\Re$ satisfies the following properties:

(i) $\Re$ is a convex set.
(ii) $\Re$ is a closed set.
(iii) $\forall (V, E) \in \Re, (-V', E') \geq 0$ and $(-V', E') \leq (-V, E) \Rightarrow (V', E') \in \Re$.

Proof.
(i) From equation (6), one obtains immediately

$$\Re = \{(V', E') \in R_+^2; \exists x \in \Im, (-V', E') \leq (-V(R(x)), E(R(x)))\}.$$

Assume that (V_1, E_1) and $(V_2, E_2) \in \Re$. Thus, one can deduce that there exists $x^1, x^2 \in \Im$ such that

$$(-V_1, E_1) \leq (-V(R(x^1)), E(R(x^1)))$$

and

$$(-V_2, E_2) \leq (-V(R(x^2)), E(R(x^2))) \in \Re.$$

Let us show that

$$\theta(V_1, E_1) + (1-\theta)(V_2, E_2) \in \Re, \qquad \forall \theta \in [0, 1].$$

Since $V(R(\cdot))$ is a convex function, one gets immediately the inequality

$$\theta V_1 + (1-\theta)V_2 \geq \theta(V(R(x^1))) + (1-\theta)(V(R(x^2))) \geq V(R(\theta x^1 + (1-\theta)x^2)).$$

Moreover, we have

$$\theta E_1 + (1-\theta)E_2 \leq E(R(\theta x^1 + (1-\theta)x^2)).$$

Thus, since

$$\{x \in R^n; Ax \leq b, \sum_{i=1,\dots,n} x_i = 1, x_i \geq 0\}$$

is a convex set, there exists

$$x = \theta x^1 + (1-\theta)x^2 \in \Im$$

such that

$$(-V(R(x)), E(R(x))) \geq \theta(-V^1, E^1) + (1-\theta)(-V^2, E^2).$$

From the expression (6), this implies

$$\theta(-V^1, E^1) + (1-\theta)(-V^2, E^2) \in \Re$$

and (i) is proven.

(ii) The functions $V(R(\cdot))$ and $E(R(\cdot))$ are continuous with respect to x; thus, $\aleph$ is a closed set. Using the result in Briec and Lesourd (Ref. 31), we get that $\{\aleph + (R_+ \times (-R_+))\}$ is closed and obviously (ii) holds.

(iii) From equation (6), the fact that

$$\forall (V, E) \in \aleph, (-V', E') \leq (-V, E) \Rightarrow (V', E') \in \Re$$

and (iii) can be deduced. □

From the above properties of the representation set, it is possible now to define the notion of an efficiency measure in the specific context of the Markowitz portfolio theory. Before introducing our own approach, existing efficiency measures in the context of portfolio benchmarking are briefly reviewed.

The first measure, introduced by Morey and Morey (Ref. 21), computes the maximum expansion of the mean return while the risk is fixed at its current level [this also seems the approach taken by Sengupta (Ref. 20)].

From our definition of the representation set, this mean-return expansion function is defined by

$$D_{MRE}(x) = \sup\{\theta; (V(R(x)), \theta E(R(x))) \in \Re\}. \tag{9}$$

In a similar vein, the same authors define a risk contraction function as follows:

$$D_{RC}(x) = \inf\{\lambda; (\lambda V(R(x)), E(R(x))) \in \Re\}. \tag{10}$$

This function measures the maximum proportionate reduction of risk while fixing the mean-return level. These authors apply these functions to measure investment fund performance.

Now, the shortage function [Luenberger (Ref. 14)] is introduced and its properties are studied in the context of the Markowitz portfolio theory. It is shown below (see Proposition 3.2) that it encompasses the functions (9) and (10) as special cases. To achieve this objective, we introduce the efficiency improvement possibility (EIP) function defined as follows.

Definition 3.1. The function defined as

$$S_g(x) = \sup\{\delta; (V(R(x)) - \delta g_V, E(R(x)) + \delta g_E) \in \Re\}$$

is the EIP function for the portfolio x in the direction of vector $g = (-g_V, g_E)$.

Notice that the EIP function is very similar to the directional distance function, another name for the shortage function introduced in production analysis by Chambers, Chung, and Färe (Ref. 13). The directional distance function looks for simultaneous changes in the direction of reducing inputs x and expanding outputs y; i.e., $g = (-g_x, g_y)$.

The principle of the EIP function is illustrated in Figure 2. The EIP function looks for improvements in the direction of both an increased mean return and a reduced risk. For instance, the inefficient portfolio A is projected onto the efficient frontier at point B.

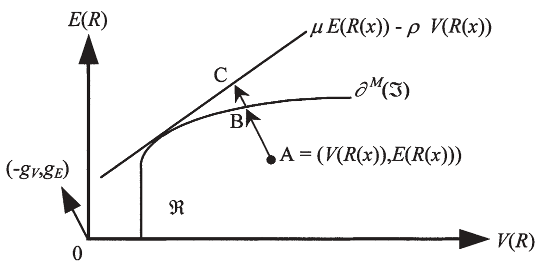

Fig. 2. Efficiency improvement possibility function and decomposition.

The pertinence of the EIP function as a portfolio management efficiency indicator results from some of its elementary properties, as summarized in the following proposition.

Proposition 3.2. Let S_g be the EIP function defined on $\Im$. S_g has the following properties:

(i) $x \in \Im \Rightarrow S_g(x) < +\infty$.
(ii) If $(g_V, g_E) > 0$, then $S_g(x) = 0 \Leftrightarrow x \in \partial^M(\Im)$ (weak efficiency).
(iii) $\forall x, y \in \Im, (-V(R(y)), E(R(y))) \leq (-V(R(x)), E(R(x))) \Rightarrow S_g(x) \leq S_g(y)$ (weak monotonicity on $\aleph$).
(iv) S_g is continuous on $\Im$.
(v) If $g_V = -V(R(x))$ and $g_R = 0$, then $D_{RC}(x) = 1 - S_g(x)$.
(vi) If $(g_V, g_E) > 0$ and $g_R = E(R(x))$, then $D_{MRE}(x) = 1 + S_g(x)$.

Proof.
(i) From the definition of the representation set, if $x \in \Im$, then the subset

$$C(x) = \{(V', E') \in \Re; (V', -E') \leq (V(R(x)), -E(R(x)))\}$$

is bounded. It follows trivially that $S_g(x) < +\infty$.
(ii) Assume that $x \notin \Lambda^M(\Im)$. In such a case, there exists some $(V', E') \in \Re$ such that

$$(-V', E') \geq (-V(R(x)), E(R(x))).$$

But, from Definition 3.1, it follows immediately that

$$S_g(x) > 0.$$

Consequently, it can be deduced that

$$S_g(x) = 0 \Rightarrow x \in \Lambda^M(\Im).$$

To prove the converse, let

$$(V(R(x)) - S_g(x)g_V, E(R(x)) + S_g(x)g_E).$$

Assume that $S_g(x) > 0$. Since $(g_V, g_E) > 0$, we get

$$(-V(R(x)) + S_g(x)g_V, E(R(x)) + S_g(x)g_E) > (-V(R(x)), E(R(x))).$$

It can be deduced immediately that $x \notin \Lambda^M(\Im)$ and (ii) holds.
(iii) This follows from Luenberger (Ref. 14).
(iv) Let the function $T: \Re \rightarrow R_+$ be defined by

$$T(V, E) = \sup\{\delta; (V - \delta g_V, E + \delta g_E) \in \Re\}.$$

Since $\Re$ is convex and satisfies the free disposal rule, it is easy to show the continuity of T. Moreover, since mean and variance are continuous functions with respect to x, (iv) holds.

(v) and (vi) result from making some obvious changes [see e.g. Chambers, Chung, and Färe (Ref. 13)]. □

Briefly commenting on these properties, the use of the EIP function guarantees only weak efficiency. It does not exclude projections on vertical parts of the frontier allowing for an additional expansion in terms of the expected return. Furthermore, portfolios with weakly dominated risk and return characteristics are classified only as weakly less efficient. Finally, the last two parts establish clearly a link with the Morey and Morey (Ref. 21) single-dimension efficiency measurement orientations in (9) and (10). Implementing some obvious changes, a simple proof for these links is straightforwardly derived for instance from Chambers, Chung, and Färe (Ref. 13). Section 4 studies the EIP function from a duality standpoint.

4. Duality, Shadow Risk Aversion, and Mean-Variance Utility

Markowitz (Ref. 29) conceived portfolio selection as a two-step procedure, whereby the reconstruction of the efficient set of portfolios in a first step is followed subsequently by picking the optimal portfolio for a given preference structure. To provide a dual interpretation of the EIP function, the indirect mean-variance utility function must be defined first [see e.g. Farrar (Ref. 2) or Philippatos (Ref. 7)].

Definition 4.1. For given parameters (ρ, μ), the function defined as

$$\begin{aligned} U^*(\mu, \rho) = \sup \quad & \mu E(R(x)) - \rho V(R(x)), \\ \text{s.t.} \quad & Ax \leq b, \\ & \sum_{i=1,\dots,n} x_i = 1, \quad x \geq 0, \end{aligned}$$

is called the indirect mean-variance utility function.

Therefore, the maximum value function for the decision maker is simply determined for a given set of parameters (ρ, μ) representing his or her risk aversion. Knowledge of these parameters allows selecting a unique efficient portfolio among those on the weakly efficient frontier maximizing the decision maker direct mean-variance utility function. Furthermore, Farrar (Ref. 2) suggested to trace the set of efficient portfolios by solving this dual problem for different sets of parameters (ρ, μ).

More elaborate dual frameworks exist in the literature. For instance, Varian (Ref. 19) describes nonparametric test procedures verifying whether or not a suitable mean-variance utility function rationalizes the observed portfolio choices and asset prices. This contribution adheres to the previously mentioned tradition and does not depend on asset price information.

To grasp duality in our framework, it is useful to distinguish between overall, allocative, and portfolio efficiency when evaluating the scope for improvements in portfolio management. The following definition clearly distinguishes between these concepts.

Definition 4.2. Let S_g be the EIP function defined on $\Im$.

(i) The overall efficiency (OE) index is the quantity

$$\mathrm{OE}(x,\rho,\mu) = \sup\{\delta; \mu(E(R(x)) + \delta g_E) - \rho(V(R(x)) - \delta g_V) \leq U^*(\rho,\mu)\}.$$

(ii) The allocative efficiency (AE) index is the quantity

$$\mathrm{AE}(x,\rho,\mu) = \mathrm{OE}(x,\rho,\mu) - S_g(x).$$

(iii) The portfolio efficiency (PE) index is the quantity

$$\mathrm{PE}(x) = S_g(x).$$

This definition implies immediately

$$\mathrm{OE}(x,\rho,\mu) = [U^*(\rho,\mu) - U_{(\rho,\mu)}(x)]/(\rho g_V + \mu g_E). \tag{11}$$

Thus, overall efficiency (OE) is simply the ratio between (a) the difference between (maximum) indirect mean-variance utility (Definition 4.1) and the value of the direct mean-variance utility function for the observation evaluated and (b) the normalized value of the direction vector $g = (-g_V, g_E)$ for the given parameters (ρ,μ).

Expanding on the decomposition introduced in Definition 4.2, portfolio efficiency (PE) guarantees only reaching a point on the portfolio frontier, not necessarily a point on the frontier maximizing the investor indirect mean-variance utility function. In this sense, it is similar to the notion of technical efficiency in production theory. Allocative efficiency (AE), by contrast, measures the needed portfolio reallocation, along the portfolio frontier, to achieve the maximum of the indirect mean-variance utility function. This requires adjusting an eventual portfolio efficient portfolio in function of relative prices, that is, the parameters of the mean-variance utility function. Overall efficiency ensures that both these ideals are achieved simultaneously.

Obviously, the following additive decomposition identity holds:

$$\mathrm{OE}(x, \rho, \mu) = \mathrm{AE}(x, \rho, \mu) + \mathrm{PE}(x) \tag{12}$$

Notice that changes in the risk-aversion parameters (ρ, μ) alter the slope of the indirect utility function. While the amount of PE is invariant to these changes, the relative importance of AE and OE normally changes.

In Figure 2, this decomposition is illustrated for a portfolio denoted by point A. For simplicity, assume that

$$\|g\| = \|(-g_V, g_E)\| = 1,$$

where $\|\cdot\|$ is the usual Euclidean metric. In terms of this figure, it is easy to see that

$$\mathrm{OE} = \|\mathrm{C} - \mathrm{A}\|, \qquad \mathrm{PE} = \|\mathrm{B} - \mathrm{A}\|, \qquad \mathrm{AE} = \|\mathrm{C} - \mathrm{B}\|.$$

The indirect mean-variance utility function turns out to be a useful tool to characterize the representation set $\Re$. In particular, by using duality, one can state the following property.

Proposition 4.1. The representation set $\Re$ admits the following dual characterization:

$$\Re = \{(V, E) \in R^2; \mu E - \rho V \leq U^*(\rho, \mu)\} \cap R^2_+.$$

Proof. By definition,

$$\Re = \{\aleph + (R_+ \times (-R_+))\} \cap R^2_+.$$

However, if $(\rho, \mu) \notin R^2_+$, then

$$\sup\{U_{(\rho,\mu)}(x); (V(R(x)), E(R(x))) \in \aleph + (R_+ \times (-R_+))\} = +\infty.$$

Since for any mean-variance vector, we have

$$(V, E) \in \aleph + (R_+ \times (-R_+)),$$

it can be deduced that

$$U^*(\rho, \mu) \geq \mu E - \rho V.$$

Now, assume that

$$(V, E) \notin \Re.$$

From Proposition 3.1, $\Re$ is convex. From the separation theorem, there exists $(\rho, \mu) \in R^2_+$ such that

$$\mu E - \rho V > U^*(\rho, \mu).$$

Consequently,

$$U^*(\rho, \mu) \geq \mu E - \rho V$$

implies

$$(V, E) \in \Re$$

and Proposition 4.1 follows. □

Proposition 4.1 prepares the link between the shortage function and the indirect utility function in Proposition 4.2. In particular, the duality result in Proposition 4.2 shows that the EIP function can be derived from the indirect mean-variance utility function, and conversely. It is inspired by Luenberger (Ref. 14), who established duality between the expenditure function and the shortage function.

Proposition 4.2. Let S_g be the EIP function defined on $\Im$. S_g has the following properties:

(i) $S_g(x) = \inf\{U^*(\rho, \mu) - U_{(\rho,\mu)}(x);\ \mu g_E + \rho g_V = 1,\ \mu \geq 0,\ \rho \geq 0\}$.

(ii) $U^*(\rho, \mu) = \sup\{U_{(\rho,\mu)}(x) - S_g(x);\ x \in \Im\}$.

Proof. The proof is a straightforward consequence of Luenberger (Ref. 14). □

This result proves that the EIP function can be computed over the dual of the mean-variance space. The support function of the representation set is the indirect utility function U^*.

Attention turns now to studying the properties of the EIP function that presume differentiability at the point where the function is evaluated. Therefore, the following adjusted risk aversion function is introduced:

$$(\rho, \mu)(x) = \operatorname{argmin}\{U^*(\rho, \mu) - U_{(\rho,\mu)}(x); \mu g_E + \rho g_V = 1, \mu \geq 0, \rho \geq 0\}, \quad (13)$$

that characterizes implicitly the agent risk aversion. It could also be labeled a shadow indirect mean-variance utility function, since it adopts a reverse approach by searching for the parameters (ρ, μ) defining a shadow risk aversion that renders the current portfolio optimal for the investor. For these, the parameters (ρ, μ) are such that OE = PE, since AE = 0 by definition. This function is similar to the adjusted price function defined by Luenberger (Ref. 14) in consumer theory, hence its naming as the adjusted risk aversion function.

Proposition 4.3. Let S_g be the EIP function defined on $\Im$. At the point where S_g is differentiable, it has the following properties:

(i) $\partial S_g(x)/\partial x = \partial U_{(\rho,\mu)(x)}(x)/\partial x = (\mu(x)I - 2\rho(x)\Omega)R$,

(ii) $\partial S_g(x)/\partial V(R(x))|_{E(R(x))=Ct} = \rho(x)$,
$\partial S_g(x)/\partial E(R(x))|_{V(R(x))=Ct} = -\mu(x)$,

where R denotes the vector of expected asset returns and I is a unit vector of appropriate dimensions.

Proof.
(i) The proof is obtained by the standard envelope theorem. The relationship

$$\partial S_g(x)/\partial x = \partial U_{(\rho,\mu)(x)}(x)/\partial x$$

is obvious. Since

$$\partial U_{(\rho,\mu)(x)}(x)/\partial x = \mu(x)R - 2\rho(x)\Omega R,$$

the result can be deduced.
(ii) The proof for (ii) is obtained in a similar way. □

Result (i) shows that the variations of the shortage function with respect to x are identical to the variation of the indirect utility function, but calculated with respect to the adjusted risk aversion function. Moreover, it can be linked directly to the return of each asset and the covariance matrix. Furthermore, result (ii) shows that the shortage function decreases when the expected return increases.

As shown below, there is a link between the adjusted risk aversion function and some kind of Marshallian demand for each asset. First, introduce the matrix of derivatives

$$[B]_{i,j} = \begin{bmatrix} \partial\rho/\partial x \\ \partial\mu/\partial x \end{bmatrix}_{i,j}. \tag{14}$$

Moreover, given a risk aversion vector (ρ, μ), the Marshallian demand for assets is defined by

$$m(\rho,\mu) = \text{argmax}\{U_{(\rho,\mu)(x)}(x); x \in \Im\}. \tag{15}$$

This allows the definition of some kind of Slutsky matrix,

$$[S]_{i,j} = [\partial m(\rho,\mu)/\partial\rho, \partial m(\rho,\mu)/\partial\mu]_{i,j}. \tag{16}$$

As shown in the next proposition, this Slutsky matrix can be linked to the matrix B.

Proposition 4.4. Let S_g be the EIP function defined on $\Im$. At the point where S_g is differentiable, it has the following properties:

(i) $BS = \left\{[1/(\rho g_V + \mu g_E)]I - [1/(\rho g_V + \mu g_E)^2]\begin{pmatrix}\rho\\ \mu\end{pmatrix} \times (g_V, g_E)\right\};$

(ii) $S^T B^T = \left\{[1/(\rho g_V + \mu g_E)]I - [1/(\rho g_V + \mu g_E)^2]\begin{pmatrix}g_V\\ g_E\end{pmatrix} \times (\rho, \mu)\right\};$

(iii) $BB^+ = I - [1/(\rho g_V + \mu g_E)^2]\begin{pmatrix}g_V\\ g_E\end{pmatrix} \times (g_V, g_E).$

Proof.

(i) Consider

$$(\bar{\rho}, \bar{\mu}) = (\rho, \mu)/(\rho g_V + \mu g_E).$$

There are the equalities:

$$\begin{aligned}\partial\bar{\rho}/\partial\rho &= \sum_{k=1,\dots,n} (\partial\bar{\rho}/\partial x_k)(\partial m_k/\partial\rho)\\ &= 1/(\rho g_V + \mu g_E) - \rho g_V/(\rho g_V + \mu g_E)^2,\end{aligned}$$

$$\begin{aligned}\partial\bar{\mu}/\partial\mu &= \sum_{k=1,\dots,n} (\partial\bar{\mu}/\partial x_k)(\partial m_k/\partial\mu)\\ &= 1/(\rho g_V + \mu g_E) - \mu g_E/(\rho g_V + \mu g_E)^2,\end{aligned}$$

$$\begin{aligned}\partial\bar{\rho}/\partial\mu &= \sum_{k=1,\dots,n} (\partial\bar{\rho}/\partial x_k)(\partial m_k/\partial\rho)\\ &= -\rho g_E/(\rho g_V + \mu g_E)^2,\end{aligned}$$

$$\begin{aligned}\partial\bar{\mu}/\partial\rho &= \sum_{k=1,\dots,n} (\partial\bar{\rho}/\partial x_k)(\partial m_k/\partial\rho)\\ &= -\mu g_V/(\rho g_V + \mu g_E)^2.\end{aligned}$$

Now, since

$$BS = \begin{pmatrix} \sum_{k=1,\dots,n} (\partial\bar{\rho}/\partial x_k)(\partial m_k/\partial\rho) & \sum_{k=1,\dots,n} (\partial\bar{\rho}/\partial x_k)(\partial m_k/\partial\rho) \\ \sum_{k=1,\dots,n} (\partial\bar{\mu}/\partial x_k)(\partial m_k/\partial\rho) & \sum_{k=1,\dots,n} (\partial\bar{\mu}/\partial x_k)(\partial m_k/\partial\mu) \end{pmatrix},$$

the result can be deduced.

(ii) This is obtained by taking the transpose of (i).
(iii) This follows by combining (i) and (ii). □

This proof can also be derived from Luenberger (Ref. 32). This result states that the Slutsky matrix, characterizing the Marshallian demand for each asset, is a type of skewed pseudo-inverse of the matrix B.

5. Computational Aspects of the EIP Function

The representation set $\Re$, defined by expression (6), can be used directly to compute the EIP function by using standard quadratic optimization methods. Assume a sample of m portfolios or investment funds $y^1, y^2, \ldots, y^m$. Now, consider a specific portfolio y^k for $k \in \{1, \ldots, m\}$ whose performance needs to be gauged. The shortage function for this portfolio y^k under evaluation is computed by solving the following quadratic program:

$$\begin{array}{lll} \text{(P1)} & \max & \delta, \\ & \text{s.t.} & E(R(y^k)) + \delta g_E \leq E(R(x)), \\ & & V(R(y^k)) - \delta g_V \geq V(R(x)), \\ & & Ax \leq b, \\ & & \sum_{i=1,\ldots,n} x_i = 1, \qquad x_i \geq 0, \quad i = 1, \ldots, n. \end{array}$$

From equations (2) and (3), program (P1) can be rewritten as follows:

$$\begin{array}{lll} \text{(P2)} & \max & \delta, \\ & \text{s.t.} & E(R(y^k)) + \delta g_E \leq \sum_{i=1,\ldots,n} x_i E(R_i), \\ & & V(R(y^k)) - \delta g_V \geq \sum_{i,j} \Omega_{i,j} x_i x_j, \\ & & Ax \leq b, \\ & & \sum_{i=1,\ldots,n} x_i = 1, \qquad x_i \geq 0, \quad i = 1, \ldots, n. \end{array}$$

To assess its performance, one quadratic program is solved for each portfolio. To obtain the entire decomposition from Definition 4.2, the only requirement is to compute the additional quadratic program from Definition 4.1. Then, applying expression (11) and Definition 4.2 itself, the components OE and AE follow suit.

All of the above programs can be seen as special cases of the following standard form:

$$\begin{array}{lll} \text{(P3)} & \min & c^T z, \\ & \text{s.t.} & L_j(z) = \alpha_j, \qquad j = 1, \ldots, q, \\ & & Q_k(z) \leq \beta_k, \qquad k = 1, \ldots, r, \\ & & z \in R^p, \end{array}$$

where L_j is a linear map for $j = 1, \ldots, q$ and Q_k is a positive semidefinite quadratic form for $k = 1, \ldots, r$. In the case of program (P2), $q = 1$ and $r = n + 3$, the latter because there are n nonnegativity constraints. Program (P3) is a standard quadratic optimization problem [see Fiacco and McCormick (Ref. 33), Luenberger (Ref. 34)].

A novel result of some practical significance is that the adjusted risk aversion function (13) can be derived from the Kuhn-Tucker multipliers in program (P2). This is shown in the next proposition.

Proposition 5.1. Let $k \in \{1, \ldots, m\}$ be such that program (P2) has a regular optimal solution. Let $\lambda_E \geq 0$ and $\lambda_V \geq 0$ be respectively the Kuhn-Tucker multipliers of the first two constraints in program (P2). If the EIP function is differentiable at point $y^k \in \Im$, then this yields

(i) $$\partial S_g(y)/\partial V(R(y))\Big|_{\substack{y=y^k \\ E(R(y))=E(R(y^k))}} = \lambda_V,$$

$$\partial S_g(y)/\partial E(R(y))\Big|_{\substack{y=y^k \\ V(R(y))=V(R(y^k))}} = -\lambda_E.$$

(ii) The adjusted price function is identical to the Kuhn-Tucker multipliers:

$$(\rho, \mu)(y^k) = (\lambda_V, \lambda_E).$$

Proof.

(i) The proof is based on the sensitivity theorem [e.g. Luenberger (Ref. 34)]. A solution of program (P2) is obtained immediately by solving the program

$$\begin{aligned}
\text{(P4)} \quad \min \quad & -\delta, \\
\text{s.t.} \quad & -\sum_{i=1,\ldots,n} x_i E(R_i) + \delta g_E \leq -E(R(y^k)), \\
& \sum_{i,j} \Omega_{i,j} x_i x_j + \delta g_V \leq V(R(y^k)), \\
& Ax \leq b, \\
& \sum_{i=1,\ldots,n} x_i = 1, \ -x_i \leq 0, \quad i = 1, \ldots, n.
\end{aligned}$$

Remark that all the constraint functions on the left-hand side in the two first inequalities are convex. Therefore, program (P4) has the standard form described in Luenberger (Ref. 34).

Now, consider the parametric program

$$\begin{aligned}
\text{(P5)} \quad \min \quad & -\delta, \\
\text{s.t.} \quad & -\sum_{i=1,\ldots,n} x_i E(R_i) + \delta g_E \le c_E, \\
& \sum_{i,j} \Omega_{i,j} x_i x_j + \delta g_V \le c_V, \\
& Ax \le b, \\
& \sum_{i=1,\ldots,n} x_i = 1, \qquad x_i \ge 0, \quad i = 1, \ldots, n.
\end{aligned}$$

Since program (P2) has a regular optimal solution, the bordered Hessian of program (P4) at the optimum is nonsingular. Consequently, the sensitivity theorem applies. Let $x^*(c_V, c_E)$ be the optimal solution of the parametric program (P5). Let $-\delta^*(x^*(c_V, c_E))$ denote the corresponding optimal value function. By definition, the Kuhn-Tucker multipliers of programs (P2) and (P4) are identical. From the sensitivity theorem, we have

$$\partial(-\delta^*(x^*(c_V, c_E)))/\partial c_V|_{c_V = V(R(y^k))} = -\lambda_V,$$
$$\partial(-\delta^*(x^*(c_V, c_E)))/\partial c_E|_{c_E = -E(R(y^k))} = -\lambda_E.$$

We deduce immediately that

$$\partial S_g(y)/\partial V(R(y))\Big|_{\substack{y=y^k \\ E(R(y))=E(R(y^k))}} = -\partial(-\delta^*(x^*(c_V, c_E)))/\partial c_V|_{c_V = V(R(y^k))} = \lambda_V.$$

Moreover,

$$\begin{aligned}
&\partial S_g(y)/\partial E(R(y))\Big|_{\substack{y=y^k \\ V(R(y))=V(R(y^k))}} \\
&= -\partial(-\delta^*(x^*(c_V, -c_E)))/\partial(-c_E)|_{-c_E = E(R(y^k))} \\
&= \partial(-\delta^*(x^*(c_V, c_E)))/\partial c_E|_{c_E = -E(R(y^k))} = -\lambda_E.
\end{aligned}$$

This ends the proof of part (i).

(ii) This result is immediate from Proposition 4.3 (ii). □

The interest of this approach based on quadratic programming concerns not only the original Markowitz model with short sales excluded. Of course, when short sales are not excluded or when there exists a riskless asset with zero variance and nonzero positive return, then the efficient frontier is determined by simpler, analytical solutions without recourse to quadratic optimization [e.g. Elton, Gruber, and Padberg (Ref. 30)]. However, in general, the quadratic programming approach remains valid. In particular, since quadratic program (P2) can be derived from (P3), it does not require a positive-definite covariance matrix. Therefore, the models remain equally valid under these

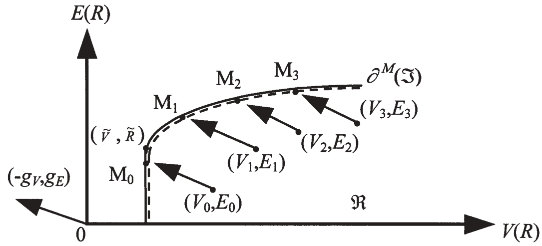

Fig. 3. Portfolio efficiency analysis: Projections onto nonparametric frontier.

cases, with practical applications to measuring asset management efficiency for e.g. regulated funds of futures and unregulated hedge funds.

Figure 3 illustrates this logic behind the performance gauging of portfolios using program (P2). Computation of these quadratic programs provides an inner bound approximation (dashed line) of the true, unknown portfolio frontier (solid line). This envelopment frontier is akin to the concept of production frontiers. This estimator is a nonparametric method, inasmuch as no functional form is specified for the Pareto frontier. The technically inefficient observations (V_0, E_0) to (V_3, E_3) are evaluated and projected onto the portfolio frontier using the same direction vector g. By adding fictitious points or by implementing a critical line search following Markowitz (Ref. 29), it is possible to refine the approximation of the efficient set of portfolios until it coincides with the Markowitz frontier.

Two general remarks conclude this section: the first is concerned with the possibility of weakly efficient portfolios; the second focuses on the selection of a direction vector $g = (-g_V, g_E)$ in all these mathematical programs.

First, the projection of (V_0, E_0) onto a vertical segment of the set of weakly efficient portfolios illustrates the scope for further removing inefficiencies until the global minimum-variance portfolio is reached. A pragmatic solution is to substitute the global minimum-variance portfolio, that provides a better expected return for the same risk, for projection points representing weakly efficient portfolios (identifiable by positive slack variables). Theoretical solutions that could be developed require sharpening the definition of the efficient frontier, or formulating doubts about the choice of direction $g = (-g_V, g_E)$ for weakly efficient portfolios (e.g. selecting a direction that guarantees at least a projection onto the global minimum-variance portfolio). Such developments are beyond the scope of this contribution. Furthermore, assuming an interest in estimating the OE decomposition (OE implying strongly efficient portfolios), the problem of the weakly efficient portfolios is limited to the PE component and leads only to a slight change in the relative importance of both components (AE versus PE).

Second, some remarks on the choice of the direction vector are useful. In principle, various alternative directions are possible [e.g. Chambers, Chung, and Färe (Ref. 13)]. For instance, it is possible to choose a common direction for all portfolios, as illustrated in Figure 3 above. This has a clear economic meaning in consumer theory where, for instance, utility may be measured using a type of distance function with respect to a common basket of goods [see the benefit function in Luenberger (Ref. 15)]. But the economic interpretation of a common direction g in production and investment theory is not evident to us.[8]

A far more straightforward choice for investment theory is to use the observation under evaluation itself, i.e.,

$$g = (-V(R(x)), E(R(x))).$$

Then, the shortage function measures the maximum percentage of risk reduction and expected return improvement. The dual formulation of the shortage function leads to a simpler interpretation,

$$\begin{aligned} S_g(x) &= \inf\{U^*(\rho,\mu) - U_{(\rho,\mu)}(x); \mu g_E + \rho g_V = 1, \mu \geq 0, \rho \geq 0\} \\ &= \inf\{U^*(\rho,\mu) - U_{(\rho,\mu)}(x); -\mu E(R(x)) + \rho V(R(x)) = 1, \mu \geq 0, \rho \geq 0\} \\ &= \inf\{U^*(\rho,\mu) - U_{(\rho,\mu)}(x); U_{(\rho,\mu)}(x) = 1, \mu \geq 0, \rho \geq 0\}. \end{aligned} \tag{17}$$

Now, using a simple normalization scheme [see Chambers, Chung, and Färe (Ref. 13)], this can be written equivalently as

$$S_g(x) = \inf\{[U^*(\rho',\mu') - U_{(\rho',\mu')}(x)]/U_{(\rho',\mu')}(x); \mu' \geq 0, \rho' \geq 0\}. \tag{18}$$

Thus, the shortage function is now interpreted as the minimum percentage improvement in the direction to reach the maximum of the utility function (i.e., the indirect utility function). Since this is conducted in the mean-variance space, the shadow risk-aversion minimizing this percentage provides a general efficiency index.

6. Empirical Illustration: Investment Funds

To show the ease of implementing the basic framework developed in this contribution, the decomposition of overall efficiency for a small sample of 26 investment funds, earlier analyzed in Morey and Morey (Ref. 21), is

[8] One possibility, suggested by P. Vanden Eeckaut, is a common direction minimizing the OE for all observations. This could prove useful when risk-aversion is unknown, and one would like to avoid penalizing observations too heavily when fixing a pair of risk-aversion parameters. Implementing this suggestion involves issues of aggregation of efficiency measures that are currently underdeveloped.

Table 1. Decomposition results for Morey and Morey (1999) sample.

Observations	OE	PE	AE	φ*
20th Century Ultra Investors	0.718	0.433	0.285	0.095
44 Wall Street Equity	0.398	0.225	0.172	0.166
AIM Aggressive Growth	0.606	0.000	0.606	0.072
AIM Constellation	0.627	0.274	0.353	0.097
Alliance Quasar A	0.616	0.550	0.066	0.205
Delaware Trend A	0.610	0.351	0.259	0.116
Evergreen Aggressive Growth A	0.742	0.538	0.204	0.108
Founders Special	0.589	0.439	0.150	0.152
Fund Manager Aggressive Growth	0.366	0.357	0.009	0.330
IDS Strategy Aggressive B	0.593	0.583	0.011	0.314
Invesco Dynamics	0.543	0.274	0.269	0.122
Keystone Amer Omega A	0.521	0.448	0.073	0.213
Keystone Small Co Growth (S-4)	0.722	0.331	0.391	0.079
Oppenheimer Target A	0.402	0.320	0.082	0.219
Pacific Horizon Aggregate Growth	0.700	0.619	0.081	0.175
PIMCo Advanced Opportunity C	0.742	0.304	0.438	0.000
Putnam Voyager A	0.541	0.323	0.218	0.135
Security Ultra A	0.559	0.503	0.057	0.225
Seligman Capital A	0.573	0.564	0.009	0.319
Smith Barney Aggregate Growth A	0.726	0.485	0.241	0.102
State St. Research Capital C	0.643	0.245	0.399	0.089
SteinRoe Capital Opport	0.588	0.317	0.272	0.116
USAA Aggressive Growth	0.708	0.545	0.162	0.128
Value Line Leveraged Growth Investment	0.481	0.319	0.163	0.161
Value Line Specific Situations	0.687	0.517	0.170	0.129
Winthrop Focus Aggregate Growth	0.026	0.014	0.011	0.332
Mean	0.578	0.380	0.198	0.162
Standard Deviation	0.155	0.159	0.152	0.087
Maximum	0.742	0.619	0.606	0.332

* Absolute risk aversion: computed via shadow prices of the EIP function (Proposition 5.1).

computed. It concerns the problem of selecting a fund of funds from a set of funds based upon the tradeoff between the expected return and risk (formally similar to composing a portfolio from a series of assets).

Return and risk are calculated over a 3-year time horizon between July 1992 and July 1995 (see Tables 1 and 8 of Ref. 21). Computing program (P2) to obtain TE, the quadratic program in Definition 4.1 for the parameters $\mu = 1$ and $\rho = 2$ to obtain the maximum of the indirect mean-variance utility function, applying the decomposition in Definition 4.2, and using (11), the results summarized in Table 1 of this paper are obtained. To save space, portfolio weights and slack variables are not reported. Risk aversion follows the conventional values for ρ that often range between 0.5 and 10 [e.g. Uysal, Trainer, and Reis (Ref. 35)].

To underline the ease of interpretation of the performance measure, the decomposition results of a single fund "44 Wall Street Equity" are commented upon. It could improve its OE by 40%, in both terms of improving its return and reducing its risk. In terms of the decomposition, 22.5% of this rather poor performance is due to PE, i.e., operating below the portfolio frontier, while 17% is due to AE, i.e., choosing a wrong mix of return and risk given the postulated risk attitudes.

The average performance of the investment funds is poor. They could improve their OE performance by about 58%, with the majority of inefficiencies being attributed to PE. Looking at the individual results, none of the investment funds suits perfectly the investors' preferences. Therefore, all are to some extent overall inefficient. The last investment fund in the list comes closest to satisfying the investors needs. Only one investment fund (number 3) is portfolio efficient and is part of the set of frontier portfolios. The residual degree of AE, listed in the third column, is small compared to the amount of PE detected. This relative importance of PE relative to AE is a common finding in production analysis. Whether or not the same general tendency holds also in portfolio gauging remains an open question. Obviously, these efficiency measures can be used easily as a rating tool.

The same results are depicted also on the graph shown in Figure 4. We plot the return and risk of investment funds in the sample, their projections onto the portfolio frontier using the shortage function (PE), and the single point on the frontier maximizing the investors' preferences (OE).

A potential major issue is the sensitivity of the results, and in particular the decomposition, to the postulated risk-aversion parameters. Using Proposition 5.1, the adjusted risk-aversion function minimizing inefficiency from the Kuhn-Tucker multipliers in program (P2) can be retrieved easily. These shadow risk aversions are shown in the last column of Table 1. Note that, for one fund, the shadow risk aversion is zero, due to positive slack in the risk dimension. For the sample, the shadow risk aversion is on average 0.162 with

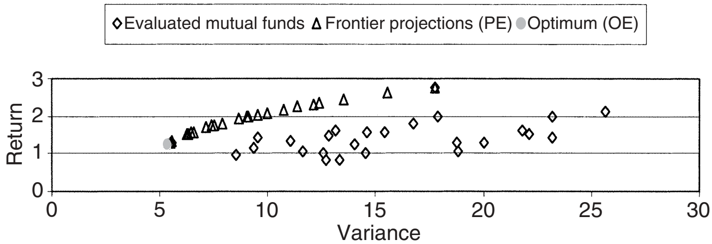

Fig. 4. Portfolio frontier: Observed portfolios and decomposition results.

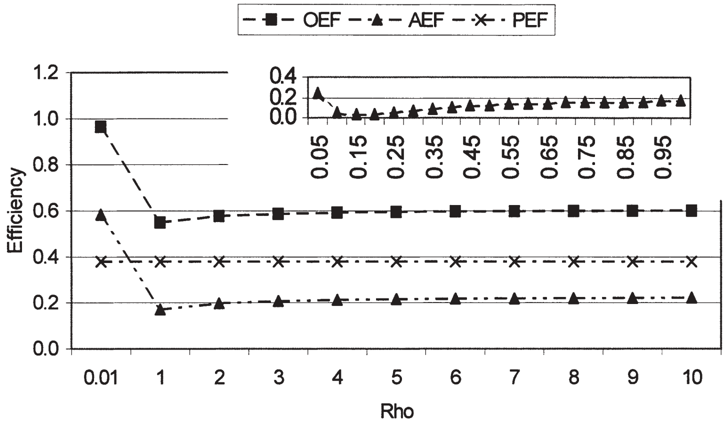

Fig. 5. Sensitivity of portfolio efficiency decomposition results for ρ.

a standard deviation of 0.087. To test the sensitivity of the decomposition results, the average efficiency components were also computed for the parameter $\mu = 1$ and a wide range of values for ρ. The results shown in Figure 5 for values ranging from almost 0 to 10 and in the detail window for the range between 0.05 and 1 indicate that the main source of inefficiency remains PE, except when risk aversion approaches zero. AE is minimized for the value of the above-mentioned shadow risk-aversion and increases slightly for deviations on both sides of this minimum.

Two remarks may be made. First, the confrontation between postulated risk-aversion parameters and shadow risk aversion could be instructive when assessing whether portfolio management strategies adhere to certain specified risk profiles. Second, the decomposition depends on a specified risk-aversion parameter, but if risk aversion is unknown, OE could equally well be ignored to focus on PE as such.

7. Conclusions

The objective of this paper has been to introduce a general method for measuring portfolio efficiency. Portfolios are benchmarked by looking simultaneously for risk contraction and mean-return augmentation using the shortage function framework [Luenberger (Ref. 14)]. The virtues of this approach can be summarized as follows: (i) it does not require the complete estimation of the efficient frontier, but approximates the true frontier by a nonparametric envelopment method; (ii) its efficiency measure lends itself perfectly for performance gauging; (iii) it yields interesting dual interpretations; (iv) it stays close to the theoretical framework of Markowitz (Ref. 29) and does not require any simplifying hypotheses. A simple empirical application on a limited sample of investment funds has illustrated the computational feasibility of this general framework.

The general idea of looking for both risk contraction and mean-return expansion is useful in a wide range of financial models. Just to mention one theoretical extension, right from the outset, alternative criteria for portfolio selection based, among others, upon higher-order moments have been developed [Philippatos (Ref. 36)]. Since the shortage function is a distance (gauge) function, a perfect representation of multidimensional choice sets, we conjecture that this framework could well be extended to these multidimensional portfolio selection approaches.

At the philosophical level, the question remains whether any eventual portfolio inefficiencies reveal judgmental errors of investors or whether these are simply the result of not accounting for additional constraints inhibiting full mean-variance efficiency. In the latter case, additional modeling efforts are required to derive the so-called fitted portfolios [Gouriéroux and Jouneau (Ref. 17)]. However, analogously with similar discussions elsewhere [Førsund, Lovell, and Schmidt (Ref. 37)], it could be conjectured that even accounting for additional constraints does not eliminate all inefficiencies. Therefore, having an unambiguous and general portfolio efficiency measure like the one proposed remains as useful as ever.

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