Vol. 81 (2004), No. 2, pp. 155–192
DOI 10.1007/s00712-003-0620-y

Journal of Economics
Zeitschrift für Nationalökonomie

Printed in Austria

Non-convex Technologies and Cost Functions: Definitions, Duality and Nonparametric Tests of Convexity

Walter Briec, Kristiaan Kerstens, and Philippe Vanden Eeckaut

Received May 6, 2002; revised version received January 3, 2003
Published online: December 23, 2003

This contribution is the first systematic attempt to develop a series of nonparametric, deterministic technologies and cost functions without maintaining convexity. Specifically, we introduce returns to scale assumptions into an existing non-convex technology and, dual to these technologies, define non-convex cost functions that are never lower than their convex counterparts. Both non-convex technologies and cost functions (total, ray-average and marginal) are characterized by closed form expressions. Furthermore, a local duality result is established between a local cost function and the input distance function. Finally, nonparametric goodness-of-fit tests for convexity are developed as a first step towards making it a statistically testable hypothesis.

Keywords: nonparametric technologies and cost functions, non-convexity, nonparametric test of convexity.

JEL classification: D24.

1 Introduction

In applied production analysis, the boundary of technology nowadays plays a prominent role in efficiency and productivity measurement (Lovell, 1993). This boundary can be estimated via several methodologies. One well-known method is the use of nonparametric, deterministic technologies and support functions. The early nonparametric test literature on production (e.g., Diewert and Parkan, 1983; Varian, 1984) focused on directly testing "revealed behavioral" conditions (e.g., Weak Axiom of Cost Minimization) that are finite in nature on a finite number of observations. Realizing that the production possibilities set is unobservable, while the producer's objective functions are, this approach leads to the determination of inner and outer approximations of technologies that contain the true but unknown frontier. More recently, the introduction of efficiency measures into these nonparametric frontier methodologies has led to extensive efficiency and productivity decompositions. For instance, while Farrell (1957) already measured technical and allocative efficiency, Färe, Grosskopf and Lovell (1983; 1985) separate technical efficiency into scale efficiencies, congestion as well as pure technical efficiencies.

A familiar result in nonparametric production analysis is that a convex monotone hull provides an inner bound approximation to the true, convex technology (Varian, 1984). Probably less known is that a non-convex monotone hull, named the Free Disposal Hull (FDH) (see Deprins, Simar and Tulkens, 1984; Tulkens, 1993), is the closest inner approximation of the true, strongly disposable (but possibly non-convex) technology so far (Färe and Li, 1998). The present paper is the first attempt to systematically extend this non-convex technology by including scaling laws, developing corresponding non-convex cost functions, and establishing a duality result between both non-convex technologies and cost functions. Duality results show that these non-convex technologies, with standard returns to scale assumptions, imply non-convex cost functions that are not lower than their convex counterparts. In addition, both types of cost functions coincide only under special conditions. While convexification is innocuous when limited to the input set, this is no longer true when extended to the input-output space. Furthermore, this work offers a framework to test for convexity. The remainder of the introduction develops the five main goals of the contribution as well as potential reasons to test for the convexity axiom in applied production analysis.

1.1 Main Goals of the Contribution

A first major goal is to extend the FDH technology by integrating traditional returns to scale assumptions without invoking convexity at all (in contrast to recent work that only partially relaxes convexity).[1] A first

1 See, e.g., Banker and Maindiratta (1986), Bogetoft (1996), Bogetoft, Tama and Tind (2000), Färe, Grosskopf and Njinkeu (1988), Kuosmanen (2001), and Post (2001). Färe, Grosskopf and Lovell (1994, pp. 52–53) link most of the different piecewise technologies presented in the literature.

key result is that this integration of returns to scale assumptions into this non-convex FDH creates the closest inner bound non-convex approximations to the true technology allowing for various scaling laws.

Since dropping convexity altogether precludes an appeal to traditional duality results, a second main goal is to develop non-convex cost functions corresponding to these non-convex technologies. These need to be compared to the traditional, empirical cost functions that impose convexity of technology, not partial convexity of the input set. This leads to a second central set of results. Traditional general convex cost functions are always lower or equal than these new, non-convex cost functions. In particular, both types of cost functions are only identical (hence convexity harmless) under certain strong assumptions: constant returns to scale and a single output. Notice that this result does not undermine any of the traditional properties of the cost function. They only refine the property that the cost function is non-decreasing in outputs: convex (non-convex) cost functions are convex (non-convex) in the outputs.

The leap from partial input convexity to convexity of technology when moving from theory to empirical application may have far reaching consequences that have so far escaped notice. Though tests for monotonicity and concavity of cost and production functions have been widely applied since the first study finding divergences between primal and dual approaches (Appelbaum, 1978), one cannot exclude that the difference between the theoretical assumption of convexity of the input set and the convexity of technology maintained in all empirical methodologies is another source of potential conflict between primal and dual results. This adds another potential problem to the list of conflicts between theory and practice in production (e.g., Love, 1999).

A third major goal is to come up with a duality result that allows inferring the original non-convex technologies from these non-convex cost functions. A third central result of this contribution is that we manage to prove a new, local duality between non-convex technologies obeying different scaling laws and the corresponding non-convex cost functions. While this local duality result is the best one can hope for in a non-convex setting, it nevertheless forms the basis for completely reconstructing the original, non-convex technologies making use of enumerative principles.

One may conjecture that the relaxation of convexity in applied production analysis has been hampered by the fear of computational complexity. Therefore, a fourth major goal of this paper is to explore this more practical issue of relevance for empirical analysis. It turns out that these non-convex nonparametric production and cost frontiers create few computational problems. In a fourth central series of results, we derive simple closed-form expressions to characterize both technology and (total, marginal and ray-average) cost functions, making use of implicit enumeration algorithms based upon vector comparisons.

A fifth and final goal is to offer a framework for testing convexity. These tests are couched in the framework of the recent efficiency measurement literature. Of course, minimal axioms are not only important when evaluating efficiency, since the volumes of technology and cost correspondence directly affect the amount of inefficiency one can reveal (Grosskopf, 1986), but are of equal importance for all traditional purposes of economic analysis. Exploiting the relationship between efficiency measures and goodness-of-fit measures used for hypothesis testing (Färe and Grosskopf, 1995; Varian, 1990), we derive nonparametric tests for the convexity hypothesis for both technologies and cost functions.

Our contribution is motivated by the conviction that empirical production analysis should build upon minimal axioms. This simply responds to a suggestion of Fuss, McFadden and Mundlak (1978, p. 223): "Given the qualitative, non-parametric nature of the fundamental axioms, this suggests […] that the more relevant tests will be non-parametric, rather than based on parametric functional forms, even very general ones." By dropping convexity altogether, the non-convex production and cost models described in this work can differentiate sharply between the effects of convexity and returns to scale assumptions on economic analysis.

In fact, the reasoning behind the impact of imposing convexity on technology on the cost function carries over to both revenue and profit functions (see also Kuosmanen, 2003). Convexity of the output set is traditionally assumed to establish duality between output distance function and revenue function. However, empirical methods imposing convexity on technology yield revenue functions that are not lower than revenue functions dropping convexity altogether. While long-run profit functions are obviously independent of convexity assumptions on technology, any other restricted profit function (e.g., short-run or expenditure-constrained profit functions) is not lower when tangent to a convex compared to a non-convex technology.

At the philosophical level, the questioning of convexity constitutes in our opinion only a first step toward examining the production axioms more seriously, both theoretically and empirically. Convexity is not a primitive axiom, but is implied by additivity and divisibility (e.g., Arrow and Hahn, 1971, p. 59–62). Apart from questioning convexity, this contribution takes a rather traditional and agnostic position. For instance, we maintain traditional global returns to scale assumptions, despite the fact that constant and non-increasing returns to scale imply divisibility. This does not imply that we accept divisibility to reject additivity.[2] While additivity is probably the least questionable assumption of both (e.g., being related to free entry), mainstream and efficiency literature often invokes the traditional global returns to scale assumptions, albeit in an instrumental way. For instance, the efficiency literature employs constant and non-increasing returns to scale technologies mainly to obtain local returns to scale information. In a similar vein, this paper appeals to traditional global returns to scale assumptions as instruments for testing the convexity hypothesis in detail.

1.2 Why Test for Convexity?

While convexity is traditionally invoked in economics, its use in production theory and in efficiency gauging in particular is questionable. We first develop two theoretical arguments. Next, we add some arguments from an empirical, statistical and managerial viewpoint based upon the FDH.

First, convexity is difficult to justify as a general property of technologies. Farrell (1959, p. 380) points to indivisibilities and economies of scale as sources of non-convexities and adds: "the onus of proof rests on those who deny their existence". Allais (1977) confirms Farrell's arguments and adds a few of his own: in particular, he favors local convexity, but rejects global convexity.[3] Shephard (1970, p. 15) interprets convexity solely in terms of time divisibility of technologies and sees no other justification for its use. His argument ignores switching costs between the underlying activities, a questionable assumption. Moreover, recent

2 There is some innovative work on technologies dropping convexity while maintaining additivity: see, for instance, the Free Replicability Hull mentioned in Bogetoft (1996), and Tulkens (1993).

3 Allais (1977, p. 188) is even more severe in his judgement when stating, "this omission [of discussing convexity] is to be found in all the contemporary literature. I do not hesitate to say that it is deliberate, for even a limited discussion of the postulate of general convexity would rapidly lead to the inevitable conclusion that this postulate cannot be accepted". Koopmans (1957) also called the widespread use of convexity in production theory a matter of analytical convenience.

theoretical developments have dispensed with convexity in deriving essential results regarding, for instance, the existence of general equilibrium (e.g., Brown, 1991). Despite this theoretical attention devoted to non-convexities in production, no general empirical methodology is available to handle these non-convexities.

In addition to this theoretical objection, there is some recent empirical evidence that non-convex costs matter in manufacturing and could explain the volatility of production relative to sales. For instance, Hall (2000), and Ramey (1991) find non-convex costs in the automobile industry due to changes in the chosen number of shifts and in the eventual shutting down of plants for a week at the time. Without entering into long-standing controversies, it seems clear that these findings are to some extent in line with arguments advanced by engineering production function advocates that engineering processes yield nicely behaved (e.g., convex) technologies only under very stringent circumstances (Wibe, 1984).

While some may question the validity of these general arguments, few would probably deny our ignorance with respect to public sector technologies in particular. Moreover, since prices are often lacking, performance gauging in the public sector is necessarily limited to technical rather than allocative efficiency. Under these circumstances, a detailed knowledge of technology is indispensable and convexity may be questionable (Bös, 1988).

Second, the at times harmless convexification of production possibility sets when developing dual value functions may have led to misinterpretations about the role of convexity. Focusing on the cost function -the main interest of this contribution- convexity is harmless if and only if convexity of the input set is concerned (e.g., Varian, 1992). But, any other convexity is not harmless. Duality implies that one can reconstruct the input set underlying the cost function. Reconstructed and original input sets are identical when the original input set is convex and monotonic. Otherwise, the convexified and monotonized reconstruction of the input set is a superset of the eventual non-convex or nonmonotonic original input set. However, both original and reconstructed input sets always have the same cost functions. Therefore, no economic information is lost when ignoring eventual non-convex or nonmonotonic parts of the technology.

However, there are two problems with imposing partial convexity (e.g., on the input set only). First, doubts about convexity in general hold a fortiori with respect to partial convexity. Second and more importantly, most empirical methodologies impose convexity of technology, not partial convexity. Imposing convexity of technology, there is no guarantee that non-convex parts of technology are irrelevant for determining minimum costs for given input prices. It turns out that non-convex minimal cost levels corresponding to non-convex technologies are never situated below the corresponding general convex cost levels. Furthermore, both types of cost functions only coincide under specific conditions. This consequence of imposing convexity of technology on empirical specifications has largely escaped notice. Henceforth, convexification of technology is not innocuous when defining technologies and cost functions, but ideally requires testing.

Though empirical production analysis is still dominated by convex technology and cost specifications, the non-convex FDH model has proven useful from empirical, statistical and managerial viewpoints.[4]

First, it contributed to empirically documenting the importance of convexity in shaping the volume of the production possibility set and the ensuing inefficiency. For example, Cummins and Zi (1998) systematically compare a wide range of parametric and nonparametric methodologies and confirm that efficiency results are consistent in terms of ranking among models of the same "family," that there can be large differences between parametric and nonparametric models, and –crucial for our argument– that nonparametric convex and FDH models differ widely: technical inefficiency is only about 2% on FDH while it amounts to about 40% on nonparametric convex technologies. This divergence is alarming when realizing that regulators recently started integrating frontier benchmarks in their policies. Last but not least, the same article also reports that FDH efficiency scores tend to correlate at least as good with conventional performance measures (e.g., return on equity) than scores estimated using nonparametric convex frontiers.

Second, FDH has attractive statistical properties. Imposing free disposability only, it is a consistent estimator for any monotone boundary, although its rate of convergence is small (Simar and Wilson, 2000). When technology is convex, which ideally requires testing, then it is possible to improve the small sample error of FDH by either using information on its asymptotic distribution of efficiency estimates, or by simulated (bootstrapped) empirical distributions. In addition, asymptotically there is no reason for imposing convexity: (i) when technology is truly convex, the FDH estimator converges to the true estimator though its convergence is

4 The terms technology and model are treated as synonyms.

notably slower than the convex estimator; (ii) while a convex model causes specification error when the true technology would be non-convex. The same arguments apply to the cost functions too.

Third, scattered in the literature, there is some evidence that managers question the validity of convexity in efficiency measurement (e.g., Epstein and Henderson, 1989). They have difficulties accepting that relative performance is determined by projections onto hypothetical piecewise linear combinations, whose feasibility cannot be observed.

Having developed these theoretical and empirical reasons for questioning convexity, this contribution unfolds as follows. Section 2 introduces basic production axioms and defines non-convex and convex nonparametric, deterministic frontier technologies. Section 3 derives non-convex total, marginal and ray-average cost functions corresponding to these technologies. Section 4 establishes and interprets a local duality result between these non-convex technologies and their corresponding cost functions and defines general, nonparametric tests for the impact of convexity conditional on a scaling law. Then, Sect. 5 introduces specific nonparametric tests for the impact of convexity on an existing taxonomy of efficiency concepts developed by Färe, Grosskopf and Lovell (1983; 1985). A brief empirical illustration is added in Sect. 6. A final section concludes and provides directions for future research.

2 Non-convex technologies: Axioms and formulations

Efficiency is measured using deterministic, nonparametric technologies based on activity analysis (see Koopmans, 1951). Production technologies are based on K observations using a vector of inputs $x \in \Re^n_+$ to produce a vector of outputs $y \in \Re^m_+$. Technology is represented by its production possibility set $T = \{(x,y): x \text{ can produce } y\}$, i.e., the set of all feasible input-output vectors. The input set $L(y)$ denotes all input vectors x producing the output vector y, i.e., $L(y) = \{x: (x,y) \in T\}$. The output set $P(x)$ is defined as the set of all output vectors y that can be obtained from the input vectors x, i.e., $P(x) = \{y: (x,y) \in T\}$. A final convenient characterization of technology for $\forall(x,y) \in T$ is the input distance function:

$$D_i(x,y) = \begin{cases} \max\{\theta\colon \theta \geq 0,\ (x/\theta, y) \in T\} & \text{if } y \neq 0\ , \\ +\infty & \text{if } y = 0\ . \end{cases} \quad (1)$$

Our contribution makes selective use of the following list of assumptions on technology:

(A.1) No free lunch ($(0,y) \in T \Leftrightarrow y = 0$); inaction is feasible ($(0,0) \in T$).
(A.2) T is closed.
(A.3) Strong or free disposal of inputs and outputs: $T = (T + N) \cap \Re_+^{n+m}$ where $N = \Re_+^n \times (-\Re_+^m)$.
(A.4) T exhibits:
  (i) Constant Returns to Scale (CRS): when $(x,y) \in T$, then $\delta(x,y) \in T$, $\forall \delta > 0$;
  (ii) Non-Increasing Returns to Scale (NIRS): when $(x,y) \in T$, then $\delta(x,y) \in T$, $\forall \delta \in [0,1]$;
  (iii) Non-Decreasing Returns to Scale (NDRS): when $(x,y) \in T$, then $\delta(x,y) \in T$, $\forall \delta \geq 1$;
  (iv) Variable Returns to Scale (VRS): when (i), (ii) and (iii) do not hold.
(A.5) T is convex.

Assumptions (A.1) and (A.2) are weak mathematical regularity assumptions. Strong or free disposability of inputs (outputs) means that inputs (outputs) can be increased (decreased) while maintaining the same output (input) levels. Specific assumptions regarding the returns to scale of technologies, i.e., the way the production process can be scaled up and down for each observation, are made in (A.4). The crucial question we focus on is the usefulness of the traditional convexity assumption (A.5). Several nonparametric technologies have been derived from these axioms.[5] The non-convex FDH satisfies (A.1) to (A.3) and (A.4 iv). Convex technologies satisfying (A.1) to (A.5) have been defined (Färe, Grosskopf and Lovell, 1985). Further, nonparametric tests focusing on (A.3) are available (e.g., Färe, Grosskopf and Lovell, 1987). This contribution focuses on developing nonparametric tests for (A.5).

We now present the non-convex technologies and contrast these with standard convex models.[6] An illuminating way to construct these production models is to start off from the production possibilities sets

5 (A.1) is sometimes ignored when defining convex nonparametric technologies (e.g., Banker, Charnes and Cooper, 1984), probably because it creates some problems for specific returns to scale assumptions. For instance, a convex technology with VRS including the origin immediately turns into a NIRS technology.

6 Convex technologies are defined in Banker, Charnes and Cooper (1984), Färe, Grosskopf and Lovell (1983; 1985), among others. FDH-based, non-convex technologies in this article have been partly outlined in Bogetoft (1996, p. 464), while Kerstens and Vanden Eeckaut (1999) mainly develop the definitions in (4).

associated with a single observation and then to build the technology of the sample as a union of sets.

Consider a set of production units $W = \{(x_1, y_1), \ldots, (x_K, y_K)\}$ that contains the null input-output vector ((0,0) ∈ W). Individual production possibilities sets are based upon one production unit (x_k,y_k), the strong disposability (*SD*) assumption and different maintained hypotheses of returns to scale (Γ):

$$\begin{aligned} S^{SD,\Gamma}(x_k, y_k) = \{(x,y)\colon x \geq \delta x_k, 0 \leq y \leq \delta y_k, \delta \in \Gamma\} \\ \text{where} \quad \Gamma \in \{VRS, CRS, NIRS, NDRS\}, \\ \text{with} \quad \text{(i) } VRS = \{\delta\colon \delta = 1\}; \\ \text{(ii) } CRS = \{\delta\colon \delta \geq 0\}; \\ \text{(iii) } NIRS = \{\delta\colon 0 \leq \delta \leq 1\}; \\ \text{(iv) } NDRS = \{\delta\colon \delta \geq 1\}. \end{aligned} \tag{2}$$

The most basic non-convex technology imposes strong disposability (A.3) and no scaling (i.e., VRS are imposed ($\delta = 1$)).[7] The other technologies add a specific assumption regarding returns to scale for each single observation. The scaling parameter δ follows the definitions in axiom (A.4). Non-convex and convex unions of these individual production possibilities sets yield the FDH and FDH-based technologies on the one hand and the traditional convex models on the other hand:

$$T^{NC,\Gamma} = \bigcup_{k=1}^{K} S^{SD,\Gamma}(x_k, y_k) \quad \text{and} \quad T^{C,\Gamma} = Co\left(\bigcup_{k=1}^{K} S^{SD,\Gamma}(x_k, y_k)\right), \tag{3}$$

where *NC* and *C* represent non-convexity and convexity, respectively, Γ is as defined in (2) and *Co*(A) denotes the convex hull of a set A. Observe that from an economic viewpoint convexity naturally requires multiple observations before it makes a difference in constructing technologies.

In addition to this approach, based on sets and their operations, the unified algebraic representation of convex technologies in Bogetoft (1996) can be extended to include non-convex technologies under different returns to scale assumptions as follows:

7 VRS technologies do not assume any particular scaling law to hold. In fact, VRS technologies satisfy NDRS and NIRS in different regions (Färe, Grosskopf and Lovell, 1994).

$$T^{\Lambda,\Gamma} = \left\{ (x,y)\colon x \geq \sum_{k=1}^{K} x_k \delta z_k, y \leq \sum_{k=1}^{K} y_k \delta z_k, z_k \in \Lambda, \delta \in \Gamma \right\},$$
$$\text{where} \quad \Lambda \in \{NC, C\},$$
$$\text{with} \quad \text{(i) } NC = \left\{ z_k \in \Re_+^K\colon \sum_{k=1}^{K} z_k = 1 \text{ and } z_k \in \{0,1\} \right\}, \tag{4}$$
$$\text{(ii) } C = \left\{ z_k \in \Re_+^K\colon \sum_{k=1}^{K} z_k = 1 \text{ and } z_k \geq 0 \right\},$$

where Γ is again as defined above.[8] There is one activity vector (z) operating subject to a non-convexity or convexity constraint and a scaling parameter (δ) allowing for a particular scaling of observations spanning the frontier.

This unified representation deviates from standard formulations of convex models in the literature to highlight the similarity between convex and non-convex production models (Färe, Grosskopf and Lovell, 1994). This formulation guarantees a one-to-one correspondence between decision variables and parameters on the one hand, and underlying production assumptions on the other hand. This advantage is important for pedagogical purposes, because convexity and returns to scale assumptions are clearly separated: (i) The scaling factor (δ) reflects the specific returns to scale assumption, (ii) Inequality signs are due to the strong disposability axioms, (iii) The sum constraint on the activity vector (z) represents the convexity hypothesis; while the same sum constraint, together with the binary integer constraint on z, represents non-convexity.

Intuitively, these non-convex technologies are the most conservative, inner bound approximations of the true technology allowing for various returns to scale hypotheses. This can be phrased in terms of the so-called minimum extrapolation principle (Banker, Charnes and Cooper, 1984; Bogetoft, 1996).

Definition 1: An empirical reference technology $T^{\Lambda,\Gamma}$, an estimate of T, satisfies the minimal extrapolation principle if $T^{\Lambda,\Gamma}$ is the smallest subset of $\Re_+^{M+N}$ containing the data W and satisfying certain technological assumptions.[9]

---

8 It is also inspired by the formulation of the convex CRS model in, e.g., Banker, Charnes and Cooper (1984, p. 1082).

9 The minimum extrapolation principle automatically guarantees (A.2).

The existence of a smallest technology is not automatically guaranteed with any set of assumptions, but needs to be proven (see Bogetoft, 1996, p. 458 for details). We are now ready to define formally that the non-convex technologies $T^{NC,\Gamma}$ satisfy the minimum extrapolation principle. This is the first central result of our contribution.

Proposition 1: The non-convex technologies $T^{NC,\Gamma}$ are the minimal extrapolation technologies containing the data $W = \{(x_1, y_1), \ldots, (x_K, y_K)\} \subset \Re_+^{M+N}$ and satisfying (A.1) to (A.4).

Proof: See the Appendix.
Evidently, the convex technologies $T^{C,\Gamma}$ are similarly the minimum extrapolation technologies satisfying (A.1) to (A.5).

Remark 1: One can define another non-convex VRS model as the intersection of non-convex NIRS and NDRS technologies: $T^{NC,VRS\text{-}\cap} = T^{NC,NIRS} \cap T^{NC,NDRS}$,[10] since $T^{NC,VRS} \subseteq T^{NC,VRS\text{-}\cap}$, $T^{NC,VRS}$ is the true VRS non-convex inner bound technology. This shows that not all technologies necessarily satisfy the minimal extrapolation principle.[11]

To solve existing convex models as well as the FDH-based technologies, we develop a mathematical programming problem based on the technology definition (4). Input efficiency ($E_i(x,y)$), i.e., the inverse of the input distance function, relative to all eight production models is computed by solving for each observation (x,y) the following binary mixed integer, nonlinear programming problem (P.1):

$$E_i(x, y) = [D_i(x, y)]^{-1} = \min\{\lambda\colon (\lambda x, y) \in T^{\Lambda,\Gamma}\}, \tag{5}$$

whereby Γ and Λ follow the definitions in (2) and (4). In the Farrell (1957) tradition, a radial efficiency measure, the inverse of the input distance

10 Obviously, $T^{NC,VRS\text{-}\cap} \subseteq T^{NC,NIRS}$ and $T^{NC,VRS\text{-}\cap} \subseteq T^{NC,NDRS}$. Following definitions of operations on technologies in Ruys (1974), $T^{NC,VRS\text{-}\cap}$ can be explicitly written as a conjunction of $T^{NC,NIRS}$ and $T^{NC,NDRS}$ technologies. The convex VRS model is similarly written as: $T^{C,VRS} = T^{C,NIRS} \cap T^{C,NDRS}$. Also both convex and non-convex CRS models can be defined as a union of NIRS and NDRS technologies. These indirect ways of estimating VRS and CRS technologies are developed in Briec et al. (2000).

11 This intersection definition also highlights the fact that in the non-convex world several VRS technologies can be defined since the VRS definition is very general, in that it only excludes three particular cases of scaling.

function, indicates the maximum amount by which inputs can be decreased while producing given outputs. $E_i(x,y)$ is bounded above by unity, which designates efficient production on the isoquant of technology.

Convex models require solving a nonlinear program, while non-convex technologies are solved using nonlinear, binary mixed integer programs. However, a simple transformation of (4) enables solving convex technologies using linear programs (Färe, Grosskopf and Lovell (1994)). Computing $E_i(x,y)$ on non-convex technologies only involves simple analytical expressions. Given the binary nature of the integers and the fact that they add up to unity, these programs can be solved using a type of implicit enumeration algorithm based upon vector comparisons (Garfinkel and Nemhauser, 1972, § 10.1). In particular, the construction of FDH-based technologies as non-convex unions of individual subsets (3) allows use of the *enumerative principle* (i.e., minimizing (maximizing) a function over a finite union of sets reduces to taking the minimum (maximum) of the minima (maxima) of the subsets).[12]

The closed-form expression for calculating $E_i(x,y)$ on FDH-based technologies intuitively consists of two main steps. (i) In the first part, a modified index set of better observations is defined allowing for a rescaling of observations in the sample according to the specific returns to scale assumption postulated. Since the vector dominance comparison accounts for the possibility that observations are rescaled within certain parameter bounds, this is coined "scaled vector dominance". The "scaled better set" $B(x,y,\Gamma)$ of observation (x,y) is conditional on a returns to scale assumption:

$$B(x,y,\Gamma) = \{(x_k,y_k): \delta x_k \le x,\ \delta y_k \ge y,\ \delta \in \Gamma\}, \tag{6}$$

where Γ characterizes returns to scale (2). Obviously, the next relation holds:

$$(x_k,y_k) \in B(x,y,\Gamma) \Leftrightarrow (x,y) \in S^{SD,\Gamma}(x_k,y_k), \tag{7}$$

where $S^{SD,\Gamma}(x_k,y_k)$ refers to the individual production possibilities sets with different returns to scale (Γ) assumptions (2). (ii) In the second part the input efficiency measure $E_i(x,y)$ is calculated given some knowledge about the scaling parameter. Instead of testing for all values of the scaling parameter (δ), for each evaluated observation one only needs to find

12 This algorithm generalizes the implicit enumeration method for FDH in the literature: e.g., Cherchye, Kuosmanen and Post (2001), or Tulkens (1993).

optimal values for δ depending on the selected orientation of measurement and the returns to scale assumption.

This intuitive procedure is condensed to a new proposition regarding the solution of the input radial efficiency measure using scaled vector dominance. However, to obtain an enumerative process for measuring $E_i(x,y)$, we first need to state precisely under which conditions (x_k,y_k) "dominates" (x,y) given Γ. To accommodate eventual null components, the following notation is introduced for any input-output vector (x, y): $I(x) = \{n \in \{1,\ldots,N\} : x_n > 0\}$ and $J(y) = \{m \in \{1,\ldots,M\} : y_m > 0\}$. We are now able to state our conditions for membership to the scaled better set without loss of generality.

Lemma 1: For $k = 1,\ldots,K$, we have the following condition:

$$(x_k, y_k) \in B(x, y, \Gamma) \Leftrightarrow \left[\max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right), \min_{n \in I(x_k)} \left(\frac{x_n}{x_{kn}}\right)\right] \cap \Gamma \neq \emptyset.$$

Proof: Condition $(x_k, y_k) \in B(x, y, \Gamma)$ can be written as $\delta x_k \leq x$, $\delta y_k \geq y$, $\delta \in \Gamma$. Equivalently, we get $\delta \min_{n \in I(x_k)} \left(\frac{x_n}{x_{kn}}\right)$ and $\delta \max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right)$ for $\delta \in \Gamma$. This immediately yields the result. □

We finally end up with the following closed-form expression for the radial input efficiency measure $E_i(x,y)$.

Proposition 2: $E_i(x,y)$ on non-convex technologies $T^{NC,\Gamma}$ is computed:[13]

$$E_i(x,y) = \begin{cases} \text{(i)} \min\limits_{(x_k,y_k) \in B(x,y,\Gamma)} \left\{ \min\limits_{n \in I(x_k)} \left(\frac{x_n}{x_{kn}}\right) \right\} \quad \text{for } \Gamma = VRS \\ \text{(ii)} \min\limits_{(x_k,y_k) \in B(x,y,\Gamma)} \left\{ \max\limits_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) \cdot \min\limits_{n \in I(x_k)} \left(\frac{x_n}{x_{kn}}\right) \right\} \\ \quad \text{for } \Gamma \in \{CRS, NIRS\} \\ \text{(iii)} \min\limits_{(x_k,y_k) \in B(x,y,\Gamma)} \left\{ \max\left(\max\limits_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right), 1 \right) \cdot \min\limits_{n \in I(x_k)} \left(\frac{x_n}{x_{kn}}\right) \right\} \\ \quad \text{for } \Gamma = NDRS. \end{cases}$$

Proof: See the Appendix.

13 No specific enumeration algorithm is developed for $T^{NC,VRS\text{-}\cap}$. As an intersection of technologies, $E_i(x,y)$ is simply the minimum of two measures: $E_i(x,y)$ computed w.r.t. $T^{NC,NIRS}$ and on $T^{NC,NDRS}$.

3 Non-convex Cost Functions: Total, Marginal and Ray-average Formulations

Except under specific conditions, when technology is non-convex, then also the cost function fails convexity. In particular, non-convex cost functions are not lower that convex ones, except in case of a single output and CRS.

It is possible to derive a cost function corresponding to these non-parametric, non-convex and convex technologies with different returns to scale assumptions. Denote the input correspondence on $T^{\Lambda,\Gamma}$ as $L^{\Lambda,\Gamma}(y) = \{x : (x, y) \in T^{\Lambda,\Gamma}\}$. Throughout the paper, it is assumed that p be a vector of positive input prices. Then, the cost function corresponding to both types of technologies is defined by:

$$C^{\Lambda,\Gamma}(p, y) = \min\{p \cdot x\colon x \in L^{\Lambda,\Gamma}(y)\}. \tag{8}$$

While convex cost functions require solving linear programming problems, we obtain the following closed-form expressions for the non-convex cost functions.

Proposition 3: Non-convex cost functions $C^{NC,\Gamma}(p, y) = \min\{p \cdot x : x \in L^{NC,\Gamma}(y)\}$ can be characterized as follows:

$$C^{NC,\Gamma}(p,y) = \begin{cases} \text{(i)} \min\limits_{k=1\ldots K} \{p \cdot x_k\colon y_k \geq y\} \quad \text{for } \Gamma = VRS, \\ \text{(ii)} \min\limits_{k=1\ldots K} \left\{ \max\limits_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) \cdot p \cdot x_k \right\} \quad \text{for } \Gamma = CRS, \\ \text{(iii)} \min\limits_{\left\{k\colon \max\limits_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) \leq 1\right\}} \left\{ \max\limits_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) \cdot p \cdot x_k \right\} \\ \quad \text{for } \Gamma = NIRS, \\ \text{(iv)} \min\limits_{k=1\ldots K} \left\{ \max\left(\max\limits_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right), 1 \right) \cdot p \cdot x_k \right\} \\ \quad \text{for } \Gamma = NDRS, \end{cases}$$

Proof: See the Appendix.

To illustrate the ease of computing the above non-convex cost function and to make a comparison with the cost function found by Hall (2000) for U.S. car manufacturing, let us consider the following example.

Example 1: For simplicity, we consider a set of data with two input-output vectors: $W = \{(1,1),(2,3)\}$, and we assume a unit input price ($p = 1$). Furthermore, to clarify our ideas, we consider the case of a NDRS technology. Using the above results, we obtain:

$$\begin{aligned} C^{NC,NDRS}(p,y) &= \min_{k=1,2}\left\{\max\left(\max\left(\frac{y}{y_k}\right),1\right)\cdot 1\cdot x_k\right\} \\ &= \min\left\{\max\left(\frac{y}{y_1},1\right)\cdot 1\cdot x_1, \max\left(\frac{y}{y_2},1\right)\cdot 1\cdot x_2\right\} \\ &= \min\left\{\max(y,1), 2\cdot\max\left(\frac{y}{3},1\right)\right\}. \end{aligned}$$

It is then clear that:

$$C^{NC,NDRS}(p,y) = \begin{cases} 1 & \text{if } 0 \le y \le 1, \\ y & \text{if } 1 \le y \le 2, \\ 2 & \text{if } 2 \le y \le 3, \\ (2/3)y & \text{if } 3 \le y < \infty. \end{cases}$$

This technology and cost function are illustrated in Figs. 1 and 2. Clearly, a glance at Fig. 4 in Hall (2000) reveals that this simple numerical example manages to reproduce the basic shape of the total cost

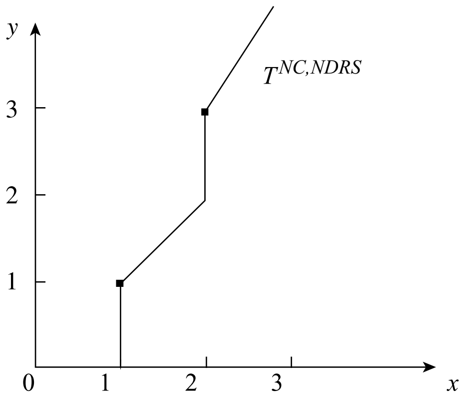

Fig. 1. A non-convex NDRS technology

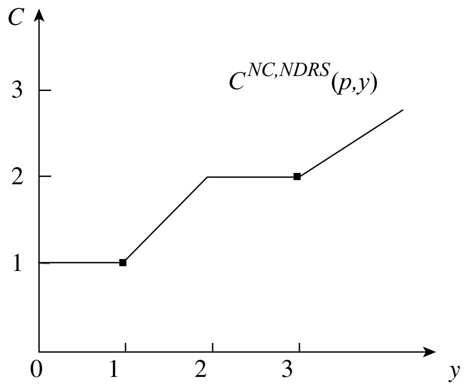

Fig. 2. A non-convex NDRS cost function

function, with one or two shifts estimated for car manufacturing. Notice that the non-convex cost function is non-decreasing in the outputs. Obviously, the properties of the cost function with respect to input prices are also preserved.

It is now important to compare this non-convex cost function to its convex counterpart. While partial convexity of the input set is innocuous, convexity of the technology is not. Trivial as it may seem, a second, so far unnoticed central result is that the cost functions based on convex technologies are always lower than or equal to cost functions based on non-convex technologies.

Proposition 4: Let the non-convex cost function $C^{NC,\Gamma}(p,y) = \min\{p \cdot x : x \in L^{NC,\Gamma}(y)\}$ and let the convex cost function $C^{C,\Gamma}(p,y) = \min\{p \cdot x : x \in L^{C,\Gamma}(y)\}$. Then, we have the following properties:

(1) In general: $C^{C,\Gamma}(p,y) \leq C^{NC,\Gamma}(p,y)$.
(2) In the case of $\Gamma = CRS$ and a single output: $C^{C,\Gamma}(p,y) = C^{NC,\Gamma}(p,y)$.

Proof: (1) Let us fix some $y \in \bigcup_{x \in \Re^n_+} P^{NC,\Gamma}(x)$. We deduce that $L^{NC,\Gamma}(y) \neq \emptyset$. Since $T^{NC,\Gamma} \subseteq T^{C,\Gamma}$, we have $L^{NC,\Gamma}(y) \subseteq L^{C,\Gamma}(y)$ and thus $L^{C,\Gamma}(y) \neq \emptyset$. Thus, $C^{C,\Gamma}(p,y)$ and $C^{NC,\Gamma}(p,y)$ are well defined. Since $L^{NC,\Gamma}(y) \subseteq L^{C,\Gamma}(y)$, we deduce that $C^{C,\Gamma}(p,y) \leq C^{NC,\Gamma}(p,y)$.
(2) Assume that the output set is one-dimensional. Consider a production technology $T^{C,CRS}$ enveloping the sample $W = \{(x_1, y_1), \ldots, (x_K, y_K)\}$.

For $k = 1, \ldots, K$, let us denote $\delta_k = \frac{y}{y_k}$. Now let us define the data-set $W' = \{(\delta_1 x_1, \delta_1 y_1), \ldots, (\delta_K x_K, \delta_K y_K)\} = \{(\delta_1 x_1, y), \ldots, (\delta_K x_K, y)\}$. Clearly, $L^{C,CRS}(y) = Co(\{\delta_1 x_1, \ldots, \delta_K x_K\}) + \Re^n_+ \quad = \{x \in \Re^n_+ : x \geq \sum_{k=1}^K z_k \delta_k x_k\}$. Since $Co(\{\delta_1 x_1, \ldots, \delta_K x_K\})$ is by definition a convex polyhedron, the minimum of a linear function is achieved by some extreme point. Therefore, $C^{C,CRS}(y) = \inf\{p \cdot x : x \in L^{C,CRS}(y)\} = \min_k\{p \cdot \delta_k \cdot x_k\} = \min_k\{p \cdot \frac{y}{y_k} \cdot x_k\}$. Now, let us calculate the cost function for a non-convex otherwise similar technology. From Proposition 3, we have:

$$C^{NC,CRS}(p,y) = \min_{k=1\ldots K}\left\{\max_{m \in J(y_k)}\left(\frac{y_m}{y_{km}}\right) \cdot p \cdot x_k\right\} = \min_{k=1\ldots K}\left\{\frac{y}{y_k} \cdot p \cdot x_k\right\} = C^{C,CRS}(p,y)$$

and the result is obtained. □

Remark 2: The analytical expression for the convex, single output, CRS cost function developed near the end of the proof (i.e., $C^{C,CRS}(p,y) = \min_{k=1\ldots K}\left\{\frac{y}{y_k} \cdot p \cdot x_k\right\}$) is – to the best of our knowledge – new.

This remarkable property that, in general, convex cost functions are never higher than non-convex cost functions has far reaching implications for the use of frontier technologies for benchmarking purposes. Imposing convex cost targets may be excessively demanding when convexity is doubtful. This is illustrated by the next example. Only for the single output CRS case convex and non-convex cost functions are identical and the convexity hypothesis cannot be tested. Notice that none of the traditional cost function properties are questioned. At best, this result refines the property that the cost function is non-decreasing in outputs: convex (non-convex) cost functions are convex (non-convex) in the outputs.

Obviously, as alluded to in the introduction, similar results could be derived for the revenue function. Revenue functions based upon convex technologies are no lower than revenue functions based upon non-convex technologies. Only in the single input CRS case, both these revenue functions coincide. Similarly, except for the long-run profit function, any other restricted profit function is no lower when tangent to a convex instead of a non-convex technology.

Example 2: Let us reconsider the set of data from Example 1. Assume again that the input price is $p = 1$ and consider again the NDRS case. Moreover, assume that $y = 2$. For the convex NDRS hull, we observe: $C^{C,NDRS}(1,2) = 3/2$. However, for the non-convex technology, we obtain a higher cost level: $C^{NC,NDRS}(1,2) = 2$.

We are now interested in deriving a closed-form expression for the corresponding non-convex marginal cost functions. Since technology is non-smooth, the cost function is non-smooth. Hence, the marginal cost function is not defined everywhere. However, differentiability of the cost function fails only at certain points and, thus, it remains almost everywhere differentiable. Therefore, it is possible to obtain a closed-form expression for the marginal cost function for any returns to scale condition on technology.

Definition 2: At points where the cost function is differentiable, the marginal cost vector is defined by:

$$C_m^{\Lambda,\Gamma}(p,y) = \frac{\partial C^{\Lambda,\Gamma}(p,y)}{\partial y}.$$

Proposition 5: Let us denote $K^{\Gamma}(y) = \arg\min_k \{p \cdot x : x \in L^{NC,\Gamma}(y)\}$ and $M(k,y) = \arg\max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right)$. The non-convex marginal cost vector $C_m^{NC,\Gamma}$ satisfies the following properties:

(1) If $\Gamma = VRS$, then the output set $P(x)$ can be partitioned in K_0 domains $D_1, D_2, \ldots, D_{K_0}$ having a non-empty interior such that $C_m^{NC,\Gamma}(p,y) = 0 \; \forall y \in \overset{\circ}{D}_k$ for $k = 1, \ldots, K_0$, where $\overset{\circ}{D}_k$ denotes the interior of the domain.

(2) If $\Gamma \in \{CRS, NIRS\}$, $\#(K^{\Gamma}(y)) = 1$ and $\#(M(k,y)) = 1$, then the cost function is differentiable, and we have:

$$\frac{\partial C^{NC,\Gamma}(p,y)}{\partial y_m} = \begin{cases} \frac{p \cdot x_{k_0}}{y_{k_0 m_0}} & \text{if } m = m_0 \\ 0 & \text{else} \end{cases} \quad \text{for } m = 1, \ldots, M$$

where $k_0 = K^{\Gamma}(y)$ and $m_0 = M(k,y)$, and # denotes the cardinality operator.

(3) If $\Gamma = NDRS$, $\#(K^{\Gamma}(y)) = 1$ and $\#(M(k,y)) = 1$, then the cost function is differentiable, and we have:

$$\frac{\partial C^{NC,\Gamma}(p,y)}{\partial y_m} = \begin{cases} \frac{p \cdot x_{k_0}}{y_{k_0 m_0}} & \text{if } \max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) > 1 \text{ and } m = m_0 \\ 0 & \text{if } \max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) < 1 \text{ or } m \neq m_0 \end{cases} \quad \text{for } m = 1, \ldots, M$$

where $k_0 = K^{\Gamma}(y)$ and $m_0 = M(k,y)$.

Proof: See the Appendix.

At points where the cost function is not differentiable, the limit of the marginal cost function is $-\infty$ and the limit of the marginal productivity is $+\infty$. The analysis of the nonparametric, convex marginal cost function in Chavas and Cox (1995) is based upon parametric programming and no analytical expressions can be obtained.

We now apply the above result to Example 1.

Example 3: We consider $W = \{(1,1),(2,3)\}$, the input price p is unity ($p = 1$), and we only consider the NDRS case. From Example 1, we obtain:

$$C_m^{NC,NDRS}(p,y) = \begin{cases} 0 & \text{if } 0 < y < 1, \\ 1 & \text{if } 1 < y < 2, \\ 0 & \text{if } 2 < y < 3, \\ (2/3) & \text{if } 3 < y < \infty. \end{cases}$$

In the context of a multi-output technology the average cost notion is undefined, but it can be replaced by the ray-average cost function (e.g., Chavas and Cox, 1999).

Definition 3: For all technologies, the ray-average cost function is defined as:

$$RAC^{\Lambda,\Gamma}(p,y) = \inf_{x,\lambda}\left\{\frac{p \cdot x}{\lambda}: x \in L^{\Lambda,\Gamma}(\lambda y), \lambda > 0\right\},$$

or in a more condensed form:

$$RAC^{\Lambda,\Gamma}(p,y) = \inf_{\lambda}\left\{\frac{C^{\Lambda,\Gamma}(p,y)}{\lambda}: \lambda > 0\right\}.$$

For non-convex technologies, the following result shows that the non-convex ray-average cost function is independent of the specific returns to scale assumption.

Proposition 6: The non-convex ray-average cost function $RAC^{NC,\Gamma}(p,y)$ satisfies the following properties:

$$RAC^{NC,\Gamma}(p,y) = \min_{k=1\ldots K}\left\{\max_{m \in J(y_k)}\left(\frac{y_m}{y_{km}}\right) \cdot p \cdot x_k\right\}$$

for $\Gamma \in \{VRS, CRS, NIRS, NDRS\}$. In particular, at the optimum we find that $\lambda^* = \left[\max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right)\right]^{-1}$ for Definition 3.

Proof: See the Appendix.

The above result provides a closed-form expression for the non-convex ray-average cost function. In contrast, the nonparametric, convex ray-average cost function requires the solution of nonlinear programming problems for each of the sample data (e.g., Chavas and Cox, 1999). Furthermore, this proposition also indicates that the non-convex ray-average cost function is independent of the returns to scale specification. In particular, it is identical to the cost function under CRS (see also Balk, 2001, p. 175 for the convex case).

4 Non-convex Cost and Distance Functions: A Local Duality Result and Nonparametric Tests

4.1 Local Duality between Non-convex Cost and Distance Functions

While cost functions for convex technologies are common knowledge, it is indispensable to provide a dual characterization for the case of the non-convex production technologies. While a duality result is hard to establish for this global cost function, a local characterization is, however, possible for each set $S^{SD,\Gamma}(x_k, y_k)$, because the latter is convex.

Definition 4: The function $C^{k,\Gamma}(p, y) = \min\{p \cdot x\colon (x, y) \in S^{SD,\Gamma}(x_k, y_k)\}$ is called the local cost function of technology $T^{NC,\Gamma}$ at point (x_k, y_k).

Lemma 2: The local cost function of technology $T^{NC,\Gamma}$ at point (x_k, y_k) is:

$$C^{k,\Gamma}(p,y) = \begin{cases} \text{(i) } p \cdot x_k & \text{if } y_k \geq y \text{ for } \Gamma = VRS, \\ \text{(ii) } \max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) \cdot p \cdot x_k & \text{for } \Gamma = CRS, \\ \text{(iii) } \max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) \cdot p \cdot x_k & \text{if } \max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right) \leq 1 \text{ for } \Gamma = NIRS, \\ \text{(iv) } \max\left(\max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right), 1\right) \cdot p \cdot x_k & \text{for } \Gamma = NDRS. \end{cases}$$

Proof: The proof is a special case of the proof established for the global cost function (see Proposition 3). □

This local cost function for a non-convex technology yields a dual relationship to the input distance function (or to the related input efficiency measure). In particular, the local cost function locally characterizes technology at point (x_k, y_k), since the subset $S^{SD,\Gamma}(x_k, y_k)$ is convex.

Proposition 7: $E_i(x, y)$ on non-convex technologies $T^{NC,\Gamma}$ is:

$$E_i(x,y) = \begin{cases} \text{(i)} \min_{(x_k,y_k)\in B(x,y,\Gamma)} \left\{ \min_{p\in\Re_+^n} \{p\cdot x_k : p\cdot x = 1\} \right\} & \text{for } \Gamma = VRS, \\ \text{(ii)} \min_{(x_k,y_k)\in B(x,y,\Gamma)} \left\{ \min_{p\in\Re_+^n} \left\{ \max_{m\in J(y_k)} \left(\frac{y_m}{y_{km}}\right) \cdot p\cdot x_k : p\cdot x = 1 \right\} \right\} & \text{for } \Gamma \in \{CRS, NIRS\}, \\ \text{(iii)} \min_{(x_k,y_k)\in B(x,y,\Gamma)} \left\{ \min_{p\in\Re_+^n} \left\{ \max\left(\max_{m\in J(y_k)} \left(\frac{y_m}{y_{km}}\right), 1 \right) \cdot p\cdot x_k : p\cdot x = 1 \right\} \right\} & \text{for } \Gamma = NDRS. \end{cases}$$

Proof: Assume that $(x_k, y_k) \in B(x, y, \Gamma)$ and express $E_i\left(x, y \middle| S^{SD,\Gamma}(x_k, y_k)\right)$ with respect to the local cost function. Since $(x_k, y_k) \in B(x, y, \Gamma)$, it follows that $(x, y) \in S^{SD,\Gamma}(x_k, y_k)$. However, since $S^{SD,\Gamma}(x_k, y_k)$ is convex, it follows that $E_i\left(x, y \middle| S^{SD,\Gamma}(x_k, y_k)\right) = \inf_{p\in\Re_+^N} \{C^{k,\Gamma}(p, y) : p \cdot x = 1\}$.

Now since $y \leq y_k$, from Lemma 2, we replace $C^{k,\Gamma}(p, y)$ by its value with respect to $\Gamma \in \{VRS, CRS, NIRS, NDRS\}$, and we obtain the result. □

In essence, this is just a traditional, convex dual relationship between local cost and input distance functions for the convex technology with a single observation $(S^{SD,\Gamma}(x_k, y_k))$. Of course, the actual technology as the union of these individual technologies can be locally non-convex around the single point on which this local duality result focuses. Consequently, the support function for the actual technology could well be nowhere near the support function of the technology with a single observation.

But, the local nature of this duality result does not prevent a straightforward reconstruction of the complete non-convex technologies with specific returns to scale assumptions from their corresponding cost functions, and vice versa. Since FDH-based technologies and cost functions are simply non-convex unions of individual subsets (3) respectively local cost functions (Definition 4), both technologies and cost functions can always be reconstructed using the enumerative principle. Clearly, a non-convex technology reconstructed on the basis of a non-convex cost function is identical to the initial non-convex technology. Since initial and reconstructed technologies have the same non-convex cost functions, no economic information is lost.

While this duality result may seem specific for the given non-convex input distance function and cost function defined, the same line of reasoning could in principle be applied to prove local duality results for other non-convex distance and cost functions. Since any non-convex technology can be reconstructed as a (finite or infinite) non-convex union of convex individual technologies, one can at best hope to establish a local duality result along the same lines. In this particular sense, the duality result is general, because it applies to the class of theoretical, non-convex technologies that are (finite or infinite) non-convex unions of convex individual technologies.

Obviously, a similar duality result could be established for non-convex output distance and revenue functions. With the exception of the long-run profit function, that is obviously independent of convexity of technology, the same line of reasoning could be applied to a non-convex directional distance function and a corresponding restricted profit function (e.g., a short-run variable profit function).[14]

4.2 General Nonparametric Tests for the Convexity of Cost and Distance Functions

In principle, the appropriateness of the convexity axiom can be tested for any comparison between convex and non-convex technologies and support functions imposing a similar returns to scale hypothesis, since differences between efficiency measures and support function levels in these components are completely attributable to convexity. Each of these can be considered a nonparametric goodness-of-fit test of the convexity axiom. To be more explicit, we define tests for the convexity of technology ($CT_i\,(x,y)$) and of the cost function ($CC_i\,(p,y)$) as ratios between the convex and non-convex input efficiency measures respectively cost functions.

---

14 Duality between profit function and directional distance function is established in Chambers, Chung and Färe (1998). The duality we allude to in the text, between a short-run directional distance function and a short run profit function, has not yet been established in a convex setting.

Definition 5: Nonparametric goodness-of-fit tests for the convexity of technologies respectively cost functions conditional on a specific scaling law Γ *are:*

(1) $CT_i^{\Gamma}(x,y) = E_i^{C,\Gamma}(x,y)/E_i^{NC,\Gamma}(x,y)$;
(2) $CC_i^{\Gamma}(p,y) = C_i^{C,\Gamma}(p,y)/C_i^{NC,\Gamma}(p,y)$.

Since $E_i^{C,\Gamma}(x,y) \leq E_i^{NC,\Gamma}(x,y)$, $0 \leq CT_i^{\Gamma}(x,y) \leq 1$. A similar reasoning applies to $CC_i^{\Gamma}(p,y)$. If $CT_i^{\Gamma}(x,y) = 1$ $(CC_i^{\Gamma}(p,y) = 1)$, then the hypothesis that technologies (cost functions) are convex cannot be rejected. In the literature so far, comparisons between traditional FDH and convex VRS production models were the only way of capturing this convexity effect. This new approach provides a perfect base to disentangle the precise impact of convexity and the returns to scale hypotheses.

5 Efficiency Decompositions and the Testing of Convexity

In the efficiency literature several taxonomies of efficiency notions have been developed (e.g., Färe, Grosskopf and Lovell, 1983; 1985; Seitz, 1971). Because it is the most widespread, in this contribution we stick to the conceptual framework developed in Färe, Grosskopf and Lovell (1983; 1985). Specific nonparametric tests for convexity are integrated into this efficiency framework.

Informally defined, Technical Efficiency (TE) requires production on the boundary of technology under the least restrictive returns to scale assumption (i.e., VRS). Production in the interior implies technically inefficiency. It is a private goal defined in terms of the best interest of the producer. Second, Overall Technical Efficiency (OTE) is always measured relative to a CRS technology, thereby conflating scale and technical efficiencies. Finally, a producer is Scale Efficient (SCE) if its size of production corresponds to a long run zero, profit competitive equilibrium configuration; it is scale inefficient otherwise. This social goal measures any divergence between the actual (VRS) and ideal (CRS) technological configuration. Overall Efficiency (OE) requires computing a cost function relative to a CRS technology with strong disposability and taking the ratio of minimal to actual costs. OE is the multiplicative result of OTE and Allocative Efficiency (AE), a residual term bridging the gap between OE and OTE. AE requires that there is no divergence between actual and optimal costs. A producer is allocatively inefficient otherwise.

The radial efficiency measure $E_i(x,y)$ used relative to different technologies entails the different concepts in this efficiency taxonomy.[15] This is reflected in the notation of $E_i(x,y)$ that can be conditioned on, e.g., a particular returns to scale hypothesis. A formal characterization of all of these notions is provided in the following definition.

Definition 6: A formal definition of input-oriented efficiency notions is provided by:

(1) Technical Efficiency $TE_i\ (x,y) = E_i(x,y|VRS)$.
(2) Overall Technical Efficiency $OTE_i\ (x,y) = E_i\ (x,y|CRS)$.
(3) Scale Efficiency $SCE_i\ (x,y) = E_i\ (x,y|CRS)\ /\ E_i\ (x,y|VRS)$.
(4) Overall Efficiency $OE_i\ (x,y,p) = C(y,p|CRS)\ /\ p{\cdot}\mathrm{x}$.
(5) Allocative Efficiency $AE_i\ (x,y,p) = OE_i\ (x,y,p)\ /\ OTE_i\ (x,y)$.

Since $E_i\ (x,y|CRS) \leq E_i\ (x,y|VRS)$, evidently $0 < SCE_i\ (x,y) \leq 1$.[16] This ratio indicates the lowest possible input combination able to produce the same output in the long run as a technically efficient combination situated on a VRS technology. It is easy to verify that all of these components are less than or equal to unity (Färe, Grosskopf and Lovell, 1994).

The embeddedness of technologies in terms of the strength of the returns to scale assumptions determines the relations between these efficiency measures. These static efficiency concepts are mutually exclusive and exhaustive and their radial measurement yields a multiplicative decomposition (Färe, Grosskopf and Lovell, 1985, pp. 188–191). Using Definition 6, the following identity readily follows:

$$OE_i(x,y,p) = AE_i(x,y,p).OTE_i(x,y), \tag{9}$$

where $OTE_i(x,y) = TE_i(x,y).SCE_i(x,y)$.

15 Radial efficiency measures project onto the isoquant and may leave some technical inefficiency unmeasured. Nonradial efficiency measures (Lovell, 1993) project onto the efficient subset of technology (see Koopmans, 1951, definition of TE) and are particularly attractive on FDH. However, we focus on radial efficiency measures for the ease of exposition.

16 For the initial proposal, see Førsund and Hjalmarsson (1974; 1979). Färe, Grosskopf and Lovell (1983) stress that technical optimal scale, and not a price-dependent (dual) notion of optimal scale, is used as the benchmark. See also Banker, Charnes and Cooper (1984).

A characteristic of production that can be further analyzed is the nature of returns to scale. For both observations on and below the frontier, it is possible to obtain qualitative information on local scale economies (i.e., for its bounding hyperplane). Since traditional methods do not apply for non-convex technologies, a more general procedure based on goodness-of-fit has been devised (Kerstens and Vanden Eeckaut, 1999).[17]

To define tests for convexity, we first clarify the relationship between convex and non-convex decompositions. As is obvious from (3), non-convex technologies are nested in their convex counterparts. As a consequence, non-convex OTE_i (x,y) and TE_i (x,y) components are larger than their convex counterparts. However, there is no a priori ordering between non-convex and convex SCE_i (x,y) components. While the underlying efficiency measures can be ordered, it is impossible to order the ratios between these efficiency measures. Non-convex cost functions never assign lower cost levels than their convex counterparts. We summarize and prove these findings in the following lemma.

Lemma 3: Relations between convex and non-convex decomposition components are:

(1) $OTE_i^C(x,y) \leq OTE_i^{NC}(x,y)$;
(2) $TE_i^C(x,y) \leq TE_i^{NC}(x,y)$;
(3) $OE_i^C(x,y,p) \leq OE_i^{NC}(x,y,p)$.

Proof: Trivial (depends on the nestedness of technologies and cost functions) and thus discarded. □

Clearly, convex technologies and cost functions may overestimate technical and overall inefficiency, making tests of the convexity hypothesis a necessity. Note that scale and allocative efficiency components cannot be ordered, because they are ratios or residuals of the other components. To be explicit, we have:

17 For inefficient observations, this characterization obviously depends on the chosen measurement orientation. Briec et al. (2000) show that the earlier comparison between CRS, NIRS and VRS models (Färe, Grosskopf and Lovell, 1983) does not apply for non-convex models, because VRS technologies are not uniquely defined (see Remark 1) and therefore no longer implicitly reveal information about NIRS and NDRS parts of technology.

$$SCE_i^C(x,y) \gtreqless SCE_i^{NC}(x,y) \quad \text{and} \quad AE_i^C(x,y) \gtreqless AE_i^{NC}(x,y). \tag{10}$$

The difference between both OTE_i (*x*,*y*) and OE_i (*x*, *y*, *p*) components can be completely attributed to convexity. Therefore, it is useful to define convexity-related technical efficiency ($CRTE_i$ (*x*,*y*)) and cost efficiency ($CRCE_i$ (*x*,*y*,*p*)) components as a ratio between these convex and non-convex components:

Definition 7: A nonparametric goodness-of-fit test for the convexity of the efficiency components based upon constant returns to scale technologies respectively cost functions is:

(1) $CRTE_i(x,y) = OTE_i^C(x,y)/OTE_i^{NC}(x,y)$;
(2) $CRCE_i(x,y,p) = OE_i^C(x,y,p)/OE_i^{NC}(x,y,p)$.

Clearly, $0 < CRTE_i\ (x,y) \leq 1$, since $OTE_i^C(x,y) \leq OTE_i^{NC}(x,y)$. A similar reasoning applies to $CRCE_i$ (*x*,*y*,*p*). When $CRTE_i$ (*x*,*y*) = 1 ($CRCE_i$ (*x*,*y*,*p*) =1), then the hypothesis that CRS technologies (cost functions) are convex cannot be rejected.

Furthermore, the definition of $CRTE_i$ (*x*,*y*) makes it possible to link non-convex and convex decompositions of OTE_i (*x*,*y*) by means of the identity:

$$OTE_i^C(x,y) = OTE_i^{NC}(x,y).CRTE_i(x,y). \tag{11}$$

The same holds true for the convexity-related cost efficiency component ($CRCE_i$ (*x*,*y*)):

$$OE_i^C(x,y,p) = OE_i^{NC}(x,y,p).CRCE_i(x,y,p). \tag{12}$$

Which of these differences between convex and non-convex decompositions proves to be most important is an empirical matter. For reasons of space, this contribution provides only a simple empirical illustration in the next section.

6 Empirical Illustration

For the empirical analysis, we partially duplicate earlier research by selecting a small sample analyzed earlier by Coelli (1996). Over the period 1953 to 1987, he analyzed the performance of the broad-acre Western Australian agricultural sector. The detailed sample data is used to construct Törnqvist quantity indices on 5 inputs ((i) livestock, (ii) materials and services, (iii) labor, (iv) capital, and (v) land) and on 3 outputs ((i) grain, (ii) sheep, and (iii) other outputs), as well as Törnqvist price indices (see Coelli, 1996, for details). Comparing the observations over the 34 years period, changes in technology are ignored. Descriptive statistics for both convex and non-convex decomposition results are presented in Table 1. The upper part of this table analyzes the technology, the lower part the cost function. To facilitate the comparison between the decompositions, we duplicate the $OTE_i(x,y)$ component in both parts of the table. To respect the multiplicative nature of these decompositions, we compute geometric averages. For reasons of space, we do not depict frequencies, but the distributions are markedly skewed to the right.

Our main findings are as follows. First, $TE_i(x,y)$ is of less importance than $SCE_i(x,y)$, whereby in the non-convex world $TE_i(x,y)$ is close to unity. Second, convexity-related technical inefficiency affects 11 obser-

Table 1. Non-convex and convex decompositions of overall efficiency

	Technology analysis						
	Non-convex decomposition				Convex decomposition		
	TE_i (x,y)	SCE_i (x,y)	OTE_i (x,y)	$CRTE_i$ (x,y)	TE_i (x,y)	SCE_i (x,y)	OTE_i (x,y)
Average[a]	0,9967	0,9120	0,9089	*0,9569*	0,9654	0,9010	0,8698
Stand. Dev.	0,0186	0,0948	0,0967	*0,0360*	0,0537	0,1052	0,1143
Minimum	0,8901	0,7301	0,7301	*0,8752*	0,8112	0,6968	0,6683
# Effic. Obs.	33	14	14	*11*	22	11	11
	Cost function analysis						
	Non-convex decomposition				Convex decomposition		
	OTE_i (x,y)	AE_i (x,y)	OE_i (x,y)	$CRCE_i$ (x,y)	OTE_i (x,y)	AE_i (x,y)	OE_i (x,y)
Average[a]	0,9089	0,8447	0,7678	*0,9776*	0,8698	0,8629	0,7506
Stand. Dev.	0,0967	0,0909	0,1470	*0,0239*	0,1143	0,0884	0,1548
Minimum	0,7301	0,6737	0,5357	*0,9381*	0,6683	0,6741	0,5047
# Effic. Obs.	14	5	5	*15*	11	5	5

[a] Geometric average

vations and accounts for about 4.2% on average and 12.5% at most. Third, allocative inefficiencies related to the cost function dominate all other sources of ill performance and both decompositions come up with somewhat similar percentages. Fourth, convexity-related cost inefficiency affects 15 observations and is only about 2.2% on average and 6.2% at worst. Finally, the bottom row reveals that the number of efficient observations per component is in the non-convex case obviously greater or equal to the numbers obtained in the convex case.

Without attributing too much weight to this empirical analysis based on a small sample, we conclude that convexity seems to make a difference on the average level as well as on individual observations. In particular, the nonparametric tests reject the convexity axiom for about a third of the observations. Hence, practitioners should be aware of the potential impact of convexity on performance gauging. Obviously, more extensive empirical analyses are called for.

7 Conclusions and Directions for Future Research

Starting from an existing non-convex production model (FDH), several nonparametric deterministic technologies have been explored introducing various returns to scale assumptions. The corresponding non-convex cost functions have also been derived. A key result is that non-convex cost functions are never lower than convex cost functions. This result refines the property that cost functions are non-decreasing in outputs: while convex cost functions are convex in the outputs, non-convex cost functions are non-convex in outputs.

This contribution has obtained analytical solutions to characterize both non-convex input distance functions (or their inverse, the input efficiency measure) and total cost functions. In addition, closed-form expressions have equally been obtained for the marginal and ray-average cost functions. This obviously opens up a wide range of possibilities for their empirical application. Furthermore, a local duality result has been established between input distance functions and the corresponding "local" cost functions. The resulting series of non-convex technologies and cost functions yield a decomposition of overall efficiency ($OE_i\ (x,y,p)$) that is similar to the existing one based on convex models. The formal relations between convex and non-convex decompositions and their respective technical, scale and allocative efficiency components have been spelled out in detail. In general, the use of convex technologies and cost functions overestimates both technical and overall inefficiencies. A hitherto unnoticed result worth stressing is that convex and non-convex cost functions are in general not identical, except in the particular case of single output CRS models. Some simple numerical examples and a small empirical application show the tractability of the approach and reveal the potentially different results that may emerge as a consequence of imposing convexity.

Our study was limited to input distance functions and cost functions. As pointed out in the text, one obvious extension to our work is to derive similar results from the revenue and short-run profit perspectives. Empirical methodologies imposing convexity on technology, indeed, yield revenue functions that are not lower than revenue functions without convexity, while any restricted profit function (e.g., due to short-run input or output fixity, expenditure-constraints, etc.) is not lower when tangent to a convex compared to a non-convex technology. The traditional argument about the long-run profit functions being independent of convexity remains valid, but it turns out to be the exception rather than the rule (see also Kuosmanen, 2003).

In addition, three further methodological extensions may seem worthwhile to pursue in the future. First, having illustrated the importance of dispensing with convexity for technical, scale and overall efficiency measurement, it is worthwhile to enlarge the range of non-convex technologies such that the congestion component could also be evaluated. This would complete the development of a static non-convex efficiency decomposition. Second, recently Simar and Wilson (2002) have developed proper statistical tests for nonparametric frontier models, though they have limited themselves so far to testing the global scale behavior of technologies. Extending their tests to the convexity hypothesis would be most valuable. Third, when panel data are available, it is obviously possible to employ these non-convex technologies and cost functions for a dynamic analysis of productivity change (e.g., Chavas and Cox, 1988; Diewert and Parkan, 1983). For instance, it would be interesting to investigate the effect of using non-convex instead of convex technologies when computing Malmquist productivity indices.[18]

---

18 Productivity measures based on FDH have been applied in Tulkens and Malnero (1996). Furthermore, this new non-convex decomposition can be integrated into any of the available decompositions of the Malmquist productivity index (Balk, 2001; Färe, Grosskopf and Lovell, 1994).

Overall, this methodological development should make people more cautious about invoking the convexity assumption in performance gauging. In particular, since convex costs are less than, or equal to, non-convex costs, imposing the former for benchmarking purposes may be unrealistic when convexity is in doubt. We hope these new technologies and cost functions, as well as the resulting efficiency decomposition, prove useful in enlarging the methodological choices open to practitioners. Lacking proper statistical tests when comparing specifications, it is important that practitioners have a precise idea of the effect of each assumption. Using the relation between efficiency measures and goodness-of-fit tests, our convexity related efficiency component provides exactly such a tool.

Appendix

Proof of Proposition 1: First, it is clear that $T^{NC,\Gamma}$ contains W and satisfies (A.1)–(A.4). We need to prove that for any technology T containing $W = \{(x_1, y_1), \ldots, (x_k, y_k)\} \subset \Re_+^{M+N}$ and satisfying (A.1) to (A.4), we have $T^{NC,\Gamma} \subset T$. Assume that $(x,y) \in T^{NC,\Gamma}$. Then, by definition, there exists some $(x_k, y_k) \in W$ such that $(x, y) \in S^{SD,\Gamma}(x_k, y_k)$. Consequently, there are some $(\bar{x}_k, \bar{y}_k) \in S^{SD,\Gamma}$ and some $\delta \in \Gamma$, such that $(x_k, y_k) = \delta(\bar{x}_k, \bar{y}_k)$. But, since T contains W and satisfies (A.1) to (A.4), then $(x_k, y_k) \in T$. Consequently, $T^{\Lambda,\Gamma} \subset T$ and this terminates the proof. □

Proof of Proposition 2: (i) By definition, we have $E_i\left(x, y \middle| S^{SD,\Gamma}(x_k, y_k)\right) = \min\{\lambda : \lambda x \geq x_k, y_k \geq y\} = \min\left\{\lambda : \lambda \geq \max_{n \in I(x)} \left(\frac{x_{kn}}{x_n}\right), y_k \geq y\right\}$. However, since $x \geq x_k$, clearly $x_{kn} > 0 \Rightarrow x_n > 0$. Thus, $\max_{n \in I(x)} \left(\frac{x_{kn}}{x_n}\right) = \min_{n \in I(x_k)} \left(\frac{x_n}{x_{kn}}\right)$. Consequently, $E_i\left(x, y \middle| S^{SD,\Gamma}(x_k, y_k)\right) = \min\left\{\lambda : \lambda \geq \min_{n \in I(x_k)} \left(\frac{x_n}{x_{kn}}\right), y_k \geq y\right\}$ and we obtain the result. For parts (ii) and (iii), first, assume that $(x_k, y_k) \in B(x, y, \Gamma)$ and calculate $E_i\left(x, y \middle| S^{SD,\Gamma}(x_k, y_k)\right)$. We have $\delta y_k \geq y, \delta \in \Gamma$. This implies $\delta \geq \max_{m \in J(y_k)} \left(\frac{y_m}{y_{km}}\right)$ for $\delta \in \Gamma$. We immediately deduce:

$$E_i\big(x,y\big|S^{SD,\Gamma}(x_k,y_k)\big)=\min\left\{\lambda:\lambda x\geq\delta x_k,\delta\geq\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right),\delta\in\Gamma\right\}$$
$$=\min\left\{\lambda:\lambda\geq\max_{n\in I(x_k)}\left(\frac{x_{kn}}{x_n}\right)\cdot\delta,\delta\geq\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right),\delta\in\Gamma\right\}.$$

Now, there are two cases: (1) Under CRS ($\Gamma = \Re_+$) or NIRS ($\Gamma = [0,1]$), the lower bound of the set $\left[\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right), \min_{n\in I(x_k)}\left(\frac{x_n}{x_{kn}}\right)\right] \cap \Gamma$ is necessarily $\delta_{\min} = \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)$. Consequently, we obtain $E_i\big(x,y\big|S^{SD,\Gamma}(x_k,y_k)\big) = \max_{n\in I(x_k)}\left(\frac{x_{kn}}{x_n}\right)\cdot\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)$ and from the enumerative principle, this concludes the proof. (2) Under NDRS ($\Gamma = [1,+\infty[$), the lower bound is $\delta_{\min} = \max\left(\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right), 1\right)$. In particular, $\delta_{\min} = 1$ if $\left[\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right), \min_{n\in I(x_k)}\left(\frac{x_n}{x_{kn}}\right)\right] \not\subset [1,+\infty[$. Hence, $E_i\big(x,y\big|S^{SD,\Gamma}(x_k,y_k)\big) = \max_{n\in I(x_k)}\left(\frac{x_{kn}}{x_n}\right)\cdot\max\left(\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right), 1\right)$. This terminates the proof from the enumerative principle. □

Proof of Proposition 3: (i) If $\Gamma = \mathit{VRS}$, then $L^{NC,\Gamma}(y) = \{x : (x,y) \in T^{NC,\Gamma}\} = \bigcup_{y_k\geq y}\{x \in \Re^N_+ : (x,y) \in S^{SD,\Gamma}(x_k,y_k)\}$. But since $\min\{p\cdot x : (x,y) \in S^{SD,\Gamma}(x_k,y_k)\} = p\cdot x_k$, the result follows. (ii) Under CRS: $L^{NC,\Gamma}(y) = \bigcup_k\{x \in \Re^N_+ : (x,y) \in S^{SD,\Gamma}(x_k,y_k)\}$. The cost function is then given by: $\min\{p\cdot x : (x,y) \in S^{SD,\Gamma}(x_k,y_k)\} = \min\{p\cdot x : x \geq \delta x_k, y \leq \delta y_k\} = \min\left\{p\cdot x : x \geq \delta x_k, \delta \geq \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\right\} = \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\cdot p\cdot x_k$ and the result follows. (iii) If $\Gamma = \mathit{NIRS}$, then the minimum cost is achieved by some x_k if there exists some $\delta \in [0,1]$ such that $\delta y_k \geq y \Leftrightarrow \delta \geq \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)$. The existence of such a δ satisfying $\delta \geq \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)$ and $\delta \leq 1$ implies that $\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right) \leq 1$. Then, we deduce that $L^{NC,\Gamma}(y) = \bigcup_{\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\leq 1}\{x \in \Re^N_+ : (x,y) \in S^{SD,\Gamma}(x_k,y_k)\}$.

Therefore, the cost function over $S^{SD,\Gamma}(x_k, y_k)$ is given by: $\min\{p\cdot x : (x,y)\in S^{SD,\Gamma}(x_k,y_k)\} = \min\{p\cdot x : x\geq \delta x_k, y\leq \delta y_k, \delta\in[0,1]\} = \min\left\{p\cdot x : x\geq \delta x_k, \delta\geq \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\right\} = \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\cdot p\cdot x_k$ and the result is proven. (iv) If $\Gamma = NDRS$, then minimum cost is achieved by some x_k if there exists some $\delta\geq 1$, such that $\delta y_k\geq y \Leftrightarrow \delta \geqslant \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)$. This condition always holds. Consequently, we deduce that $L^{NC,\Gamma}(y) = \bigcup_k\{x\in\Re^N_+ : (x,y)\in S^{SD,\Gamma}(x_k,y_k)\}$. The cost function over the set $S^{SD,\Gamma}(x_k,y_k)$ is therefore given by: $\min\{p\cdot x : (x,y)\in S^{SD,\Gamma}(x_k,y_k)\} = \min\{p\cdot x : x\geq \delta x_k, y\leq \delta y_k, \delta\geq 1\} = \min\left\{p\cdot x : x\geq \delta x_k, \delta\geq \max\left(\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right), 1\right)\right\} = \max\left(\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right), 1\right)\cdot p\cdot x_k$ and the result follows. □

Proof of Proposition 5: (1) Let the set $K(y) = \{k\in\{1,\ldots,K\} : y_k\geq y\}$. Let the function $F : y\to F(y) = \#(K(y))$. Let $\mathrm{Im}(F) = \{F(y) : y\in\Re^M_+\}$. It is immediate that $\mathrm{Im}(F)\subset\{1,\ldots,K\}$. Let us denote $K_0 = \#(\mathrm{Im}(F))$ Thus, $\{y\in\Re^M_+\} = \bigcup_{k\in\mathrm{Im}(F)} F^{-1}(k)$. It is then clear that for each $y\in F^{-1}(k)$, we have $F(y) = k$ is constant. Ranging the subset $F^{-1}(k)$ from 1 to K_0, we obtain the partition of the output set. Moreover, it is easy to show that each D_k has a non-empty interior $\overset{\circ}{D}_k$. Consequently, for each interior point y there is a neighborhood $N(y,\varepsilon)\subset\overset{\circ}{D}_k$ such that F is constant. Thus, F is constant over $\overset{\circ}{D}_k$. However, when F is a constant, then $K(y) = \{k\in\{1,\ldots,K\} : y_k\geq y\}$ stays the same on the interior of D_k and $C^{NC,\Gamma}(p,y) = \min_{k\in K(y)} p\cdot x_k$ is also constant. This implies that the derivative is defined and $C^{NC,\Gamma}_m = 0$ on the interior $\overset{\circ}{D}_k$. (2) To prove this result we need an intermediate result. Assume there are k differentiable functions $f_1, f_2, \ldots, f_k$. Assume there is some i_0, such that $i_0 = \arg\min_i\{f_i\}$. In this case, $f_{i_0}(x) = \min_i\{f_i\}$ is differentiable. Note that differentiability is not guaranteed if $\#\left(\arg\min_i\{f_i\}\right)\geq 2$. Now, under conditions $\#\left(K^\Gamma(y)\right) = 1$ and $\#(M(k,y)) = 1$, we obtain:

$C^{NC,\Gamma}(p,y) = \min_{k=1,\dots,K}\left\{\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\cdot p\cdot x_k\right\} = \left(\frac{y_{m_0}}{y_{k_0 m_0}}\right)\cdot p\cdot x_{k_0}$. Thus, we end up with the marginal cost function $\frac{\partial C^{NC,\Gamma}(p,y)}{\partial y_m} = \begin{cases}\frac{p\cdot x_{k_0}}{y_{k_0 m_0}} \text{ if } m = m_0\\ 0 \quad \text{else}\end{cases}$.

(3) The proof is obtained in a similar way. 4) If $k_0 = \{K^\Gamma(y)\}$ and $m_0 = \{M(k,y)\}$, then we have $C^{NC,\Gamma}(p,y) = \max\left(\frac{y_{m_0}}{y_{k_0 m_0}}, 1\right)\cdot p\cdot x_k$, and by enumeration of the two cases the result follows. □

Proof of Proposition 6: First, it is clear that for any $y \in \Re^M_+$ and $k = 1,\dots,K$, there exists some $\lambda > 0$ and some $x \in \Re^N_+$ such that $\left(x, \frac{y}{\lambda}\right) \in S^{SD,\Gamma}(x_k, y_k)$ (because we can choose λ sufficiently small). It follows that:

$$RAC^{NC,\Gamma}(p,y) = \inf_{x,\lambda}\left\{\frac{p\cdot x}{\lambda} : x \in L^{NC,\Gamma}(\lambda y), \lambda > 0\right\}$$
$$= \min_{k=1,\dots,K}\left\{\inf_{x,\lambda}\left\{\frac{p\cdot x}{\lambda} : x \in S^{NC,\Gamma}(x_k, y_k), \lambda > 0\right\}\right\}.$$

Now for $\Gamma \in \{VRS, CRS, NIRS, NDRS\}$ we have $\inf_{x,\lambda}\{\frac{p\cdot x}{\lambda} : x \in S^{NC,\Gamma}(x_k, y_k), \lambda > 0\} = \inf_{x,\lambda}\{\frac{p\cdot x}{\lambda} : x \geq \delta x_k, \lambda y \leq \delta y_k, \delta \in \Gamma, \lambda > 0\}$. Letting $x' = \frac{x}{\lambda}$ we obtain $\inf_{x',\lambda}\{p\cdot x' : x' \geq \frac{\delta}{\lambda}x_k, y \leq \frac{\delta}{\lambda}y_k, \delta \in \Gamma, \lambda > 0\}$. Let us denote $A^\Gamma = \{\frac{\delta}{\lambda} : \delta \in \Gamma, \lambda > 0\}$. Now making the change $\mu = \frac{\delta}{\lambda}$ the optimization problem can be written $\inf_{x',\mu}\{p\cdot x' : x' \geq \mu x_k, y \leq \mu y_k, \mu \in A^\Gamma\}$. But, for $\Gamma \in \{VRS, CRS, NIRS, NDRS\}$, we have the inclusion $\Re_{++} = \{\mu : \mu > 0\} \subset A^\Gamma = \{\frac{\delta}{\lambda} : \delta \in \Gamma, \lambda > 0\}$. Furthermore, since at the optimum we have necessarily the condition $\mu \geq \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right) > 0$, we deduce immediately the following equalities $\inf_{x',\mu}\{p\cdot x' : x' \geq \mu x_k, y \leq \mu y_k, \mu \in A^\Gamma\} = \inf_{x',\mu}\{p\cdot x' : x' \geq \mu x_k, y \leq \mu y_k, \mu > 0\} = \inf_{x',\mu}\left\{p\cdot x' : x \geq \mu x_k, \mu \geq \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\right\} = \max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\cdot p\cdot x_k$. Also, we deduce by the enumeration principle that $RAC^{NC,\Gamma}(p,y) = \min_{k=1,\dots,K}\left\{\max_{m\in J(y_k)}\left(\frac{y_m}{y_{km}}\right)\cdot p\cdot x_k\right\}$ for $\Gamma \in \{VRS, CRS, NIRS, NDRS\}$. □

Acknowledgements

The paper benefited from comments at conference and seminar presentations in Brussels and Oviedo. Especially the comments of A. Alvarez, J.-P. Vial, L. Orea and an anonymous referee of this journal are gratefully acknowledged. The usual disclaimer applies.

References

Allais, M. (1977): "Theories of General Economic Equilibrium and Maximum Efficiency". In *Equilibrium and Disequilibrium in Economic Theory*, edited by G. Schwödiauer. Dordrecht: Reidel, pp. 129–201.

Appelbaum, E. (1978): "Testing Neoclassical Production Theory". *Journal of Applied Econometrics* 7(1): 87–102.

Arrow, K., and Hahn, F. (1971): *General Competitive Analysis*. San Francisco: Holden-Day.

Balk, B. M. (2001): "Scale Efficiency and Productivity Change". *Journal of Productivity Analysis* 15(3): 159–183.

Banker, R., Charnes, A., and Cooper, W. (1984): "Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis". *Management Science* 30(9): 1078–1092.

Banker, R., and Maindiratta, A. (1986): "Piecewise Loglinear Estimation of Efficient Production Surfaces". *Management Science* 32(1): 126–135.

Bauer, P., Berger, A., Ferrier, G., and Humphrey, D. (1998): "Consistency Conditions for Regulatory Analysis of Financial Institutions: A Comparison of Frontier Efficiency Methods". *Journal of Economics and Business* 50(2): 85–114.

Bogetoft, P. (1996): "DEA on Relaxed Convexity Assumptions". *Management Science* 42(3): 457–465.

Bogetoft, P., Tama, J. M., and Tind, J. (2000): "Convex Input and Output Projections of Non-convex Production Possibility Sets". *Management Science* 46(6): 858–869.

Bös, D. (1988): "Recent Theories on Public Enterprise Economics". *European Economic Review* 32(2–3): 409–414.

Briec, W., Kerstens, K., Leleu, H., and Vanden Eeckaut, P. (2000): "Returns to Scale on Nonparametric Deterministic Technologies: Simplifying Goodness-of-Fit Methods Using Operations on Technologies". *Journal of Productivity Analysis* 14(3): 267–274.

Brown, D. (1991): "Equilibrium Analysis with Non-convex Technologies". In *Handbook of Mathematical Economics*, vol. 4, edited by W. Hildenbrand and H. Sonnenschein. Amsterdam: North-Holland, pp. 1963–1995.

Chambers, R. G., Chung, Y., and Färe, R. (1998): "Profit, Directional Distance Functions, and Nerlovian Efficiency". *Journal of Optimization Theory and Applications* 98(2): 351–364.

Chavas, J-P., and Cox, T. (1988): "A Nonparametric Analysis of Agricultural Technology". *American Journal of Agricultural Economics* 70(2): 303–310.

Chavas, J-P., and Cox, T. (1995): "On Nonparametric Supply Response Analysis". *American Journal of Agricultural Economics* 77(1): 80–92.
Chavas, J-P., and Cox, T. (1999): "A Generalized Distance Function and the Analysis of Production Efficiency". *Southern Economic Journal* 66(2): 294–318.
Cherchye, L., Kuosmanen, T., and Post, T. (2001): "FDH Directional Distance Functions with an Application to European Commercial Banks". *Journal of Productivity Analysis* 15(3): 201–215.
Coelli, T. (1996): "Measurement of Total Factor Productivity Growth and Biases in Technological Change in Western Australian Agriculture". *Journal of Applied Econometrics* 11(1): 77–92.
Cummins, D., and Zi, H. (1998): "Comparison of Frontier Efficiency Methods: An Application to the U.S. Life Insurance Industry". *Journal of Productivity Analysis* 10(2): 131–152.
Deprins, D., Simar, L., and Tulkens, H. (1984): "Measuring Labor-efficiency in Post Offices". In *The Performance of Public Enterprises: Concepts and Measurement*, edited by M. Marchand, P. Pestieau, and H. Tulkens. Amsterdam: Elsevier, pp. 243–267.
Diewert, W., and Parkan, C. (1983): "Linear Programming Test of Regularity Conditions for Production Functions". In *Quantitative Studies on Production and Prices*, edited by W. Eichhorn, K. Neumann, and R. Shephard. Würzburg: Physica-Verlag, pp. 131–158.
Epstein, M., and Henderson, J. (1989): "Data Envelopment Analysis for Managerial Control and Diagnosis". *Decision Sciences* 20(1): 90–119.
Färe, R., and Grosskopf, S. (1995): "Nonparametric Tests of Regularity, Farrell Efficiency and Goodness-of-Fit". *Journal of Econometrics* 69(2): 415–425.
Färe, R., Grosskopf, S., and Lovell C. A. K. (1983): "The Structure of Technical Efficiency". *Scandinavian Journal of Economics* 85(2): 181–190.
Färe, R., Grosskopf, S., and Lovell, C. A. K. (1985): *The Measurement of Efficiency of Production*. Boston: Kluwer.
Färe, R., Grosskopf, S., and Lovell, C. A. K. (1987): "Nonparametric Disposability Tests". *Journal of Economics* 47(1): 77–85.
Färe, R., Grosskopf, S., and Lovell, C. A. K. (1994): *Production Frontiers*. Cambridge: Cambridge University Press.
Färe, R., Grosskopf, S., and Njinkeu, D. (1988): "On Piecewise Reference Technologies". *Management Science* 34(12): 1507–1511.
Färe, R., and Li, S-K. (1998): "Inner and Outer Approximations of Technology: A Data Envelopment Approach". *European Journal of Operational Research* 105(3): 622–625.
Farrell, M. (1957): "The Measurement of Productive Efficiency". *Journal of the Royal Statistical Society* 120A(3): 253–281.
Farrell, M. (1959): "The Convexity Assumption in the Theory of Competitive Markets". *Journal of Political Economy* 67: 377–391.
Førsund, F., and Hjalmarsson, L. (1974): "On the Measurement of Productive Efficiency". *Swedish Journal of Economics* 76(2): 141–154.
Førsund, F., and Hjalmarsson, L. (1979): "Generalized Farrell Measures of Efficiency: An Application to Milk Processing in Swedish Dairy Plants". *Economic Journal* 89(354): 294–315.

Fuss, M., McFadden, D., and Mundlak, Y. (1978): "A Survey of Functional Forms in the Economic Analysis of Production". In *Production Economics: A Dual Approach to Theory and Applications*, vol. 1, edited by M. Fuss and D. McFadden. Amsterdam: North-Holland, pp. 219–268.

Garfinkel, R., and Nemhauser, G. (1972): *Integer Programming*. New York: Wiley.

Grosskopf, S. (1986): "The Role of the Reference Technology in Measuring Productive Efficiency". *Economic Journal* 96(382): 499–513.

Hall, G. J. (2000): "Non-convex Costs and Capital Utilization: A Study of Production Scheduling at Automobile Assembly Plants". *Journal of Monetary Economics* 45(3): 681–716.

Kerstens, K., and Vanden Eeckaut, P. (1999): "Estimating Returns to Scale Using Nonparametric Deterministic Technologies: A New Method Based on Goodness-of-Fit". *European Journal of Operational Research* 113(1): 206–214.

Koopmans, T. (1951): "Analysis of Production as an Efficient Combination of Activities". In *Activity Analysis of Production and Allocation*, edited by T. Koopmans. New Haven: Yale University Press, pp. 33–97.

Koopmans, T. (1957): *Three Essays on the State of Economic Science*. New York: McGraw-Hill.

Kuosmanen, T. (2001): "DEA with Efficiency Classification Preserving Conditional Convexity". *European Journal of Operational Research* 132(2): 326–342.

Kuosmanen, T. (2003): "Duality Theory of Non-convex Technologies". *Journal of Productivity Analysis*. (forthcoming).

Love, H. A. (1999): "Conflicts between Theory and Practice in Production Economics". *American Journal of Agricultural Economics* 81(3): 696–702.

Lovell, C. A. K. (1993): "Production Frontiers and Productive Efficiency". In *The Measurement of Productive Efficiency: Techniques and Applications*, edited by H. Fried, C. A. K. Lovell, and S. Schmidt. Oxford: Oxford University Press, pp. 3–67.

Post, T. (2001): "Estimating Non-convex Production Sets – Imposing Convex Input Sets and Output Sets in Data Envelopment Analysis". *European Journal of Operational Research* 131(1): 132–142.

Ramey, V. A. (1991): "Non-convex Costs and the Behavior of Inventories". *Journal of Political Economy* 99(2): 306–334.

Ruys, P. (1974): "Production Correspondences and Convex Algebra". In *Production Theory*, edited by W. Eichhorn et al. Berlin: Springer, pp. 231–252.

Seitz, W. D. (1971): "Productive Efficiency in the Steam-Electric Generating Industry". *Journal of Political Economy* 79(4): 878–886.

Shephard, R. W. (1970): *Theory of Cost and Production Functions*. Princeton: Princeton University Press.

Simar, L., and Wilson, P. (2000): "Statistical Inference in Nonparametric Frontier Models: The State of the Art". *Journal of Productivity Analysis* 13(1): 49–78.

Simar, L., and Wilson, P. (2002): "Nonparametric Tests of Returns to Scale". *European Journal of Operational Research* 139(1): 115–132.

Tulkens, H. (1993): "On FDH Efficiency Analysis: Some Methodological Issues and Applications to Retail Banking, Courts, and Urban Transit". *Journal of Productivity Analysis* 4(1–2): 183–210.

Tulkens, H., and Malnero, A. (1996): "Nonparametric Approaches to the Assessment of the Relative Efficiency of Bank Branches". In *Sources of Productivity Growth*, edited by D. Mayes. Cambridge: Cambridge University Press, pp. 223–244.

Varian, H. (1984): "The Nonparametric Approach to Production Analysis". *Econometrica* 52(3): 579–597.

Varian, H. (1990): "Goodness-of-Fit in Optimizing Models". *Journal of Econometrics* 46(1–2): 125–140.

Varian, H. (1992): *Microeconomic Analysis*, 3rd edition. New York: Norton.

Wibe, S. (1984): "Engineering Production Functions: A Survey". *Economica* 51(204): 401–411.

Addresses of authors: – Walter Briec, JEREM, Université de Perpignan, 54 Avenue Villeneuve, F-66000 Perpignan, France (e-mail: briec@univ-perp.fr); – Kristiaan Kerstens, CNRS-LABORES, IESEG, 3 rue de la Digue, F-59800 Lille, France (e-mail: k.kerstens@ieseg.fr); – Philippe Vanden Eeckaut, GREMARS, University of Lille 3, Domaine Universitaire du "Pont de Bois", BP 149, F-59653 Villeneuve d'Ascq Cédex, France (e-mail: vdeeckaut@univ-lille3.fr).