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Returns to Scale on Nonparametric Deterministic Technologies: Simplifying Goodness-of-Fit Methods Using Operations on Technologies

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Abstract

The purpose of this short article is to simplify goodness-of-fit methods to obtain qualitative information about returns to scale for individual observations. Traditional and new goodness-of-fit methods developed for estimating returns to scale on nonparametric deterministic reference technologies are reviewed. Using composition rules for technologies with specific returns to scale assumptions, we show how these goodness-of-fit methods can be simplified in the case of convex technologies (Data Envelopment Analysis (DEA) models).

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1. Introduction

Kerstens and Vanden Eeckaut (KVE) (1999) review traditional methods for estimating returns to scale for nonparametric deterministic technologies. They propose a general method based on goodness-of-fit measures suitable for all reference technologies (including the various DEA and FDH models). For determining individual scale economies, their approach requires estimation of technologies under three returns to scale assumptions: Constant Returns to Scale (CRS), Non-Increasing Returns to Scale (NIRS), and Non-Decreasing Returns to Scale (NDRS).

In this article, we show that, conditional on convexity, NIRS and NDRS technologies suffice for retrieving all information on efficiency measurement and scale properties. For more general technologies, however, the initial results remain valid and three technologies are needed. Furthermore, we bring some clarifications in the vast literature on scale economies and prove the link between goodness-of-fit methods and the Färe, Grosskopf and Lovell (FGL) (1983, 1985, 1994) approach, the former generalising the latter.

Seiford and Zhu (1999), reviewing returns to scale methods in DEA, remark that the FGL approach has the advantage of being unaffected by alternate optima, but "the only drawback to the scale efficiency index method seems to be the requirement of three computational runs". As a consequence of our results, efficiency must only be computed under two alternative returns to scale assumptions for convex models to yield both technical and scale efficiency scores as well as the characterisation of local returns to scale. This is useful to deal with large samples or for bootstrap computations (e.g., Simar and Wilson (1998)). This simplification makes the KVE (1999) method at least as computationally parsimonious as the best of alternative methods (see Table 6 in Appa and Yue (1996) and Golany and Yu (1997)). Furthermore, given its generality, this clearly makes the goodness-of-fit method the first choice in applied analysis.

This simplification of the goodness-of-fit method builds upon results based on operations on technologies with specific returns to scale assumptions. These results are—to our knowledge—nowhere available in the literature. The next section of the paper systematically develops all results. Section 3 concludes.

2. Efficiency Measurement and Estimating Returns to Scale

2.1. Technologies and Returns to Scale: Introductory Definitions

Technology is represented by its production possibility set $T = \{(x, y) : x \text{ can produce } y\}$, based on k observations (DMUs) of inputs $x \in \mathfrak{R}^m_+$ and outputs $y \in \mathfrak{R}^n_+$. Returns to scale can be defined in terms of T. Traditionally, four assumptions on returns to scale are considered.

Definition 1. Technology $T = \{(x, y) : x \text{ can produce } y\}$ exhibits:

(i) Constant Returns to Scale (CRS) if $\delta T \subseteq \mathbf{T}, \forall \delta > 0$;

(ii) Non-Increasing Returns to Scale (NIRS) if $\delta T \subseteq T, \forall \delta \in [0, 1]$;

(iii) Non-Decreasing Returns to Scale (NDRS) if $\delta T \subseteq T, \forall \delta \geq 1$;

(iv) Variable Returns to Scale (VRS) if (i), (ii) and (iii) do not hold (in this case, only condition $\delta = 1$ is warranted).

In terms of observed production combinations, this definition, for example, requires that any observation of a CRS technology can be scaled up or down by a semi-positive scalar ($\forall (x, y) \in \mathbf{T}$ and $\delta > 0$, $\delta(x, y) \in \mathbf{T}$). It is of interest to classify technologies according to this definition.

LEMMA 1 *Let T be a production set satisfying the core Shephard axioms (see Färe (1988)):*

a) If T satisfies CRS, then T satisfies NDRS

b) If T satisfies CRS, then T satisfies NIRS

c) If T satisfies both NDRS and NIRS, then T satisfies CRS

d) If T violates both NDRS and NIRS, then T satisfies VRS.

Proof. The proof is obvious from Definition 1. ■

Definition 2. Technology T exhibits locally (see Färe (1988)):

a) Decreasing Return to Scale (DRS) if there exists some $(x_0, y_0) \in T$, $\delta_0 > 1$, $\delta_0(x_0, y_0) \notin T$.

b) Increasing Return to Scale (IRS) if there exists $(x_0, y_0) \in T$, $\delta_0 \in]0, 1]$, $\delta_0(x_0, y_0) \notin T$.

Following Definition 2, an IRS technology exhibits NDRS and not CRS; and a DRS technology exhibits NIRS but not CRS.

2.2. Estimating Returns to Scale and Operations on Technologies

Following Farrell (1957), efficiency is traditionally measured in a radial way. This paper concentrates on the radial input efficiency measure: $DF_i(x, y) = \min\{\lambda | \lambda \geq 0, (\lambda\, x, y) \in T\}$. However, the results can be easily generalised for output- and graph-oriented efficiency measurement (FGL (1985)), or to the recently defined efficiency measures based upon directional distance functions (Briec (1997), Chambers, Chung and Färe (1998)).

For estimation purpose, an operational definition of the production possibility set for a representative series of nonparametric deterministic technologies is as follows:

$$T_s = \left\{ (x, y) \,\middle|\, x \geq \delta \sum_{k=1}^{K} z_k, x_k, y \leq \delta \sum_{k=1}^{K} z_k y_k, \sum_{k=1}^{K} z_k = 1, z_k \geq 0, \delta \in \Gamma(s), \right\} \quad (1)$$

where $\Gamma(CRS) = [0, +\infty[$, $\Gamma(NIRS) = [0, 1]$, $\Gamma(NDRS) = [1, +\infty[$, and $\Gamma(VRS) = \{1\}$. This definition of T_s leads to a non-linear program when computing radial input efficiency. But substituting $\lambda_k = \delta z_k$ and some straightforward manipulations render the program linear.

For convex nonparametric deterministic technologies, it is straightforward to show the validity of the following union and intersection operations on technologies. We state this result, so far unnoticed in the literature, in Proposition 1. We denote by $\mathfrak{J}$ the set of all possible technologies satisfying the Shephard axioms.

PROPOSITION 1 *For convex nonparametric deterministic technologies:*

a) $T_{CRS} = T_{NIRS} \cup T_{NDRS}$*; and*

b) $T_{VRS} = T_{NIRS} \cap T_{NDRS}$.

Proof. First, let us prove a). From Definition 1, we have $T_{NDRS} = \cup_{\delta \geq 1} \delta T_{VRS}$. Obviously $T_{NIRS} = \cup_{0 \leq \delta \leq 1} \delta T_{VRS}$ and $T_{CRS} = \cup_{\delta \geq 0} \delta T_{VRS}$. Thus, $T \in \mathfrak{J} \Rightarrow \cup_{\delta \geq 0} \delta T = (\cup_{0 \leq \delta \leq 1} \delta T) \cup (\cup_{\delta \geq 1} \delta T)$, and $T_{CRS} = \cup_{\delta \geq 0} \delta T_{VRS} = \cup_{0 \leq \delta \leq 1} \delta T_{VRS}) \cup (\cup_{\delta \geq 1} \delta T_{VRS}) = T_{NIRS} \cup T_{NDRS}$ and a) is shown. Part b) can be proven in a similar way. From the above relationships, we deduce that: $T_{VRS} = \cup_{\delta = 1} \delta T_{VRS} = (\cup_{0 \leq \delta \leq 1} \delta T_{VRS}) \cap (\cup_{\delta \leq 1} \delta T_{VRS}) = T_{NIRS} \cap T_{NDRS}$ and b) is proven. ■

This intuitive result seems trivial, but was never given rigorous attention. Clearly, a CRS technology is the union of NIRS and NDRS technologies. Likewise, the VRS model equals, in the convex case, the intersection of NIRS and NDRS counterparts. Consequently, only two computations are needed to determine efficiency relative to all four technologies. Proposition 2 formulates this result.

PROPOSITION 2 $DF_i(x, y \mid CRS)$ *and* $DF_i(x, y \mid VRS)$ *on convex technologies can be estimated from NIRS and NDRS technologies:*

a) $DF_i(x, y \mid CRS) = \min\{DF_i(x, y \mid NIRS), DF_i(x, y \mid NDRS)\}$ *and,*

b) $DF_i(x, y \mid VRS) = \max\{DF_i(x, y \mid NIRS), DF_i(x, y \mid NDRS)\}$,

where $DF_i(x, y \mid CRS), DF_i(x, y \mid VRS), DF_i(x, y \mid NIRS)$*, and* $DF_i(x, y \mid NDRS)$ *indicate input efficiency measures computed relative to CRS, VRS, NIRS and NDRS technologies.*

Proof. The a) part follows immediately from Proposition 1. b) From (1), it is obvious that $\lambda_{\max} = \max\{DF_i(x, y \mid NIRS), DF_i(x, y \mid NDRS)\} \leq DF_i(x, y \mid VRS)$. To prove the reverse inequality, note that by definition $(DF_i(x, y \mid NIRS)x, y) \in T_{NIRS}$ and $(DF_i(x, y \mid NDRS)x, y) \in T_{NDRS}$. Since inputs are free disposable $(\lambda_{\max}x, y) \in T_{NIRS} \cap T_{NDRS}$. This implies that $\lambda_{\max} \geq DF_i(x, y \mid VRS)$ and thus equality holds and b) is proven. ■

Proposition 2 immediately allows to derive the decomposition of technical and scale efficiency (FGL (1983, 1985, 1994)), whereby scale efficiency ($SCE_i(x, y)$) is defined as the ratio between overall technical efficiency ($DF_i(x, y \mid CRS)$) and pure technical efficiency ($DF_i(x, y \mid VRS)$).

2.3. Simplifying Goodness-of-Fit Methods

Appa and Yue (1996), KVE (1999) and Seiford and Zhu (1999) review the literature on the estimation of individual DMU's returns to scale. As a brief reminder, we summarise in chronological order (year of publication) the three main methods proposed in the literature to obtain qualitative information regarding local scale economies.

First, FGL (1983) compare the components of $SCE_i(x, y)$ with a third efficiency measure evaluated on a NIRS technology. Exploiting a priori knowledge on distances between these frontiers, observations are classified in terms of returns to scale (see below). A second method uses the sum of the optimal activity vector on a CRS technology to classify the observations (Banker (1984)). The final method determines the intercept of the supporting hyperplane at the reference unit on a VRS technology (Banker, Charnes and Cooper (1984)). This amounts to a classification based upon the sign of the shadow price of the convexity constraint.

KVE (1999) proposed a more general method based on goodness-of-fit measures (Varian (1990)) suitable for all reference technologies, including the various FDH models where the second and third methods simply fail. As in the case of efficiency measurement (see §2.2), we simplify their method to test for returns to scale when using convex technologies. The following proposition states the main result of this article, simplifying Proposition 2 in KVE (1999).

PROPOSITION 3 *Using $DF_i(x, y)$ and conditional on the optimal projection point, a convex technology is locally characterised by:*

a) $IRS \Leftrightarrow DF_i(x, y \mid NDRS) = \max\{DF_i(x, y \mid NIRS), DF_i(x, y \mid NDRS)\}$;

b) $CRS \Leftrightarrow DF_i(x, y \mid NDRS) = DF_i(x, y \mid NIRS) = \max\{DF_i(x, y \mid NIRS), DF_i(x, y \mid NDRS)\}$;

c) $DRS \Leftrightarrow DF_i(x, y \mid NIRS) = \max\{DF_i(x, y \mid NIRS), DF_i(x, y \mid NDRS)\}$.

Proof. Follows directly from Proposition 2, Lemma 1 and Definition 2. ∎

Only two computations are needed to determine the returns to scale of each DMU. Computing efficiency relative to a CRS technology, mentioned in Proposition 2 of KVE (1999), is redundant for convex models.[1] This local returns to scale determination of individual DMUs remains conditional on the choice of measurement orientation. Appa and Yue (1996) independently propose a method that yields the same results. But their formulation is more complex and they state, but do not proof, a limited version of our Propositions 1 and 5.

Banker, Chang and Cooper (1996) prove equivalence of three existing methods to characterise local returns to scale, including the FGL one. Therefore, we only need to prove that the new method is equivalent to the FGL method. Using $DF_i(x, y)$ and conditional on the optimal projection point, the FGL method states that technology is characterised locally by:

$$IRS \Leftrightarrow DF_i(x, y \mid CRS) = DF_i(x, y \mid NIRS) < DF_i(x, y \mid VRS) \leq 1;$$

$$CRS \Leftrightarrow DF_i(x, y \mid CRS) = DF_i(x, y \mid NIRS) = DF_i(x, y \mid VRS) \leq 1;$$

$$DRS \Leftrightarrow DF_i(x, y \mid CRS) < DF_i(x, y \mid NIRS) = DF_i(x, y \mid VRS) \leq 1.$$

PROPOSITION 4 *The method in Proposition 3 is equivalent to the FGL method.*

Proof. As a direct consequence of Proposition 2,

a) if $DF_i(x, y \mid NIRS) = DF_i(x, y \mid NDRS)$, then $DF_i(x, y \mid CRS) = DF_i(x, y \mid VRS)$;

b) if $DF_i(x, y \mid NIRS) \neq (DF_i(x, y \mid NDRS)$, then $DF_i(x, y \mid CRS) < DF_i(x, y \mid VRS)$;

c) if $DF_i(x, y \mid NIRS) < DF_i(x, y \mid NDRS)$, then $DF_i(x, y \mid CRS) = DF_i(x, y \mid NIRS) < DF_i(x, y \mid NDRS) = DF_i(x, y \mid VRS)$;

d) if $DF_i(x, y \mid NIRS) > DF_i(x, y \mid NDRS)$, then $DF_i(x, y \mid CRS) = DF_i(x, y \mid NDRS) < DF_i(x, y \mid NIRS) = DF_i(x, y \mid VRS)$;

which clearly includes the FGL inequalities. ■

Only the a) part of Proposition 1 holds true for nonconvex technologies. Any VRS technology cannot be reconstructed from intersecting NIRS and NDRS hulls of technology. The latter construction critically depends upon the convexity hypothesis.

PROPOSITION 5 *For general, nonparametric deterministic technologies:*

a) $T \in \mathfrak{I} \Rightarrow \cup_{\delta \geq 0} \delta T = (\cup_{0 \leq \delta \leq 1} \delta T) \cup (\cup_{\delta \geq 1} \delta T)$

b) $T \in \mathfrak{I}$ *and* T *convex* $\Rightarrow T = (\cup_{0 \leq \delta \leq 1} \delta T) \cap (\cup_{\delta \geq 1} \delta T)$

Proof. Part a) is trivial, and can, e.g., follow the proof of Proposition 1. Let us now show b). Obviously, we have $T \subset ((\cup_{0 \leq \delta \leq 1} \delta T) \cap (\cup_{\delta \geq 1} \delta T))$. To prove the converse, assume that $(x, y) \in ((\cup_{0 \leq \delta \leq 1} \delta T) \cap (\cup_{\delta \geq 1} \delta T))$. In such a case there exists $\mu \in]0,1[$, and $(x', y') \in T$ such that $(x, y) = \mu(x', y')$. Moreover, there exists $\gamma \geq 1$, and $(x'', y'') \in T$ such that $\gamma(x'', y'') = (x, y) \in T$. Now, it is easy to see that (x, y) belongs to the line segment $[(x'', y''), (x', y')]$. Thus, we deduce immediately that $(x, y) = \theta(x', y') + (1 - \theta)(x'', y'')$ with $\theta = (\mu - \mu/\gamma)/(1 - \mu/\gamma)$. Since T is convex, we deduce that $(x, y) \in T$, thus $((\cup_{0 \leq \delta \leq 1} \delta T) \cap (\cup_{\delta \geq 1} \delta T)) \subset T$. Hence, the converse is proven and b) holds true. ■

The intuition is straightforward. A CRS hull is always the intersection of NIRS and NDRS hulls. But property b) does not hold without convexity.[2] Therefore, the goodness-of-fit method developed in KVE (1999) cannot, in general, be simplified.

3. Conclusion

For convex technologies, this article simplified a new, intuitive method for computing efficiency measures and estimating local returns to scale. Its advantages are that it is perfectly general (all other methods fail in this respect), and, at the practical level, that it requires only two LP's. In addition, the note has clarified the link between FGL and goodness-of-fit methods. This simplified method is useful when treating large samples or for bootstrap estimations.

We end by noting that this paper reasoned in the framework of standard LP solution algorithms found in conventional software. Recently, Hackman, Passy and Platzman (1994) offered a simple pivoting type algorithm to compute the vertices of radial two-dimensional sections of convex frontier models. When their algorithm is successfully implemented, the computational effort to obtain constant and variable returns to scale radial input- and output-oriented efficiency measures is (in their own words) "almost the same as that of solving a single LP problem" (page 162). Besides being an attractive device for graphical representations of the production possibility set, instead of focusing on projection points only, this offers a promising alternative provided their algorithm becomes more widely available.[3] However, when any type of non-radial efficiency measure is utilised, the applicability of their algorithm becomes questionable, while our own method remains valid.

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Notes

1. The same formula applies for graph-oriented efficiency measures. For output efficiency measurement, traditionally defined to be larger or equal to unity, the max operator should be replaced by a min operator. For efficiency measures based upon directional distance functions, defined to be larger or equal to zero, again a min operator is applicable.
2. For a simple example with non-convex, FDH-based technologies: see Kerstens and Vanden Eeckaut (1998).
3. We are grateful to the referee for pointing this out. Actually, Rosen, Schaffnit and Paradi (1998) use the same algorithm to derive marginal rates in DEA-type models.

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