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Interfaces with Other Disciplines

Portfolio selection with skewness: A comparison of methods and a generalized one fund result

Walter Briec [a], Kristiaan Kerstens [b,*], Ignace Van de Woestyne [c]

[a] Université de Perpignan, Perpignan, France
[b] IESEG School of Management, 3 rue de la Digue, F-59000 Lille, France
[c] Hogeschool-Universiteit Brussel, Brussels, Belgium

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ABSTRACT

This contribution compares existing and newly developed techniques for geometrically representing mean–variance–skewness portfolio frontiers based on the rather widely adapted methodology of polynomial goal programming (PGP) on the one hand and the more recent approach based on the shortage function on the other hand. Moreover, we explain the working of these different methodologies in detail and provide graphical illustrations in relation to the goal programming literature in operations research. Inspired by these illustrations, we prove two new results: a formal relation between both approaches and a generalization of the well-known one fund separation theorem from traditional mean–variance portfolio theory.

1. Introduction

The limitations of modern portfolio theory trading off risk and expected return are meanwhile well-documented. A host of empirical studies rejects the hypothesis that portfolio returns are characterized by normal distributions. Furthermore, there is ample evidence that investors care about higher moments of return distributions. A recent study evaluating the out-of-sample performance of a variety of sample-based mean–variance (MV) portfolio models designed to reduce the effect of estimation error reveals that none of these methods consistently outperforms a naive portfolio diversification rule (see DeMiguel et al., 2009).

Nevertheless, a continuous stream of new proposals aims at improving the traditional MV portfolio model: for instance, Roman et al. (2007) combine two risk measures (i.e., variance and Conditional Value-at-Risk (CVaR)) and transform this multi-objective problem into a single objective problem, or one can regularize (i.e., stabilize) the MV optimization problem by considering it as a constrained least-squares regression problem by adding a penalty term proportional to the sum of the absolute values of the portfolio weights (see Brodie et al., 2009). A host of alternative risk measures have been inserted into the traditional Markowitz bi-criteria model (e.g., semi-variance (or various other lower partial moments), mean absolute deviation, quantile shortfall risk, Gini mean difference, etc.). Part of these proposals aim at linearizing the portfolio optimization problem. Especially when portfolios face multiple additional constraints (minimal lots, transaction costs, etc.), then LP solvability is an asset (see, e.g., Mansini et al., 2003).

However, discontent with MV has alternatively led to an enormous stream of proposals to include, e.g., the skewness or (more rarely) higher order moments into portfolio theory. Limiting our discussion to mean–variance–skewness (MVS) portfolio optimization models, a variety of articles have offered alternative approaches over the years. Examples of a primal approach are found in Wang and Xia (2002) who determine MVS portfolios via a multi-objective programming approach, or in Athayde and Flôres (2004) who determine analytical solutions characterizing the MVS portfolio frontier by minimizing the variance for given mean and skewness while assuming a risk-free asset and shorting. There are plenty of other recent research lines. For example, Li et al. (2010) develop a fuzzy MVS model as well as some variations. As another instance, Konno et al. (1993) formulate a general portfolio optimization problem maximizing skewness subject to fixed expected return and variance constraints, whereby both the quadratic and cubic terms are linearly approximated to yield a mean-absolute deviation-skewness model. Note that a lot of these contributions tend to solve the MVS portfolio problem by privileging one or two of the objectives at the cost of the other(s). Starting from specifications of the indirect MVS utility function, dual approaches search for optimal portfolios via preference parameters reflecting attitudes towards risk and skewness. Jondeau and Rockinger (2006) and Harvey et al. (2010) are recent examples of such utility-based studies.

* Corresponding author. Tel.: +33 320545892; fax: +33 320574855.
E-mail addresses: briec@univ-perp.fr (W. Briec), k.kerstens@ieseg.fr (K. Kerstens), ignace.vandewoestyne@hubrussel.be (I. Van de Woestyne).

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It is fair to say that no consensus has emerged so far about a general approach to multi-moment portfolio models. While especially these primal approaches should ideally be somehow equivalent, we are unaware of any comparative study in the literature. In this context, this contribution is-to the best of our knowledge-the first attempt to develop a comparison between two primal MVS portfolio optimization approaches.

On the one hand, the seminal article by Lai (1991) develops a MVS portfolio optimization model under the assumptions of shorting and the availability of a risk-free asset (see, e.g., Chunhachinda et al., 1997 for an explicit list of assumptions). This article started a burgeoning literature whereby portfolio optimization is conceived as a multiple goal programming problem. In particular, in what is nowadays commonly referred to as the polynomial goal programming (PGP) approach, one attempts to find a compromise between several goals by minimizing some appropriate polynomial penalty function. As traditionally conceived, this results in simultaneously maximizing return and skewness for a given unit portfolio risk. This PGP methodology has become a rather popular vehicle for empirical research looking at skewness persistence in a variety of international markets (e.g., Chunhachinda et al., 1997 or Sun and Yan, 2003), emerging markets (e.g., Canela and Collazo, 2007), or other markets known to follow non-normal distributions (like commodity trading advisors (CTAs), the Collateralized Fund Obligation (CFO) Equity Tranche, hedge funds, or funds of hedge funds (FOFs): see Aboul-Enein et al. (2012), Anson (2006), Anson et al. (2007), Elkaim and Papageorgiou (2006), or Davies et al. (2009)). Leung et al. (2001) develop a method to combine several forecasts within such PGP portfolio model.

A variety of methodological refinements has been adopted to the basic Lai (1991) model: trade-offs among lower partial moments (see Chen and Shia, 2007), trade-off between return and Value-at-Risk (see Chen, 2008), index tracking (see Wu et al., 2007), among others. Furthermore, the sensitivity of results to intervalling and the nature of anualization of returns is now better understood (see Chang et al., 2008a,b). A slight generalization of the basic PGP framework is found in Leung et al. (2001). It maintains a risk-free asset but introduces a variance goal resulting in a three- (instead of two-) dimensional goal programming problem whereby the PGP objective function follows the computational form of the Minkowski distance. Davies et al. (2009) explicitly add kurtosis as a fourth objective and normalize the deviation variables.[1] These are the main empirical applications and extensions known to us. In terms of the number of publications, in our count Lai (1991) is currently the most widely applied primal MVS portfolio model around in the literature.

Obviously, multiple criteria decision analysis (MCDA) and goal programming (GP) are widely applied in the financial sector (see the survey in, e.g., Spronk et al., 2005). Discrete evaluation methods are used to assist decision-making on diverse financial problems ranging from bankruptcy and credit risk assessment over investment appraisal to portfolio selection and management, among others. The popularity of the Lai (1991) article can then be interpreted as but one exponent of the widespread development of GP models for portfolio analysis in the operational research literature. While PGP models have close connections to MCDA in general and to multiobjective optimization and GP in particular, their properties have hardly ever been analyzed in this context. Indeed, apart from Leung et al. (2001), this contribution is the first attempt to link the basic Lai (1991) model to developments in GP to clarify some confusions in the literature.

On the other hand, the shortage function approach has been developed by Briec et al. (2004) in MV and by Briec et al. (2007) in MVS, inspired by the introduction of the same function in production theory by Luenberger (1995). Basically, it provides a theoretical framework for a new approach initially proposed in the investment literature by Cantaluppi and Hug (2000) to directly measure portfolio performance by reference to its maximum potential on the (ex-ante or ex-post) portfolio frontier. Furthermore, the direct link between the shortage function and other efficiency measures transposed from the so-called Data Envelopment Analysis literature to portfolio and other financial applications is obvious (see, e.g., Joro and Na, 2006 or Lozano and Guttiérez, 2008a for portfolio applications, or Gregoriou et al. (2005) for a fund rating context).

Recently, the general case of a shortage function covering an arbitrary number of moments has been realized in Briec and Kerstens (2010). As theoretical advantage, a firm link is established between the shortage function on the one hand and mixed risk aversion utility functions (representing preferences for odd moments and dislikes for even moments) on the other hand (see Briec and Kerstens, 2010 for a general duality result).[2] One of the main practical advantages of the shortage function approach is its capability of providing geometrical representations of portfolio frontiers under a wide variety of restrictions on portfolio weights. This has been studied from a methodological angle for MVS portfolio frontiers in Kerstens et al. (2011).[3] Other empirical portfolio work based on this approach includes Jurczenko et al. (2006), Jurczenko and Yanou (2010), Lozano and Guttiérez (2008b), among others. Compared to PGP, it is an approach that makes some claim to generality in that it bridges the gap between primal and dual (i.e., utility-based) approaches (see Briec and Kerstens, 2010).

An immediate question is how these two primal approaches can be related to one another. In particular, this contribution basically attempts to answer the following pertinent questions: (i) Are both PGP and shortage function optimal points situated on the same MVS frontier? (ii) What are the mathematical relations between the PGP and shortage function approaches? (iii) Can PGP be used for reconstructing portfolio frontiers, and if so, how can one proceed? (iv) Can the PGP approach be extended beyond its initial portfolio setting, as Lai (1991) claims? In answering these four questions, we use appropriate graphical illustrations based on the shortage function and we develop new, two-dimensional geometrical portfolio representations in MVS based upon both shortage function and PGP approaches. In particular, the answers to these questions lead to two new results. Furthermore, these answers can inspire further extensions for both these two primal approaches, but also provide a basis for further investigations relating and integrating various other MVS (multi-moment) portfolio approaches.

The remainder of the paper is organized as follows. Section 2 presents the basic portfolio setting and presents the PGP technique introduced by Lai (1991) as well as the shortage function approach based on the work initiated by Briec et al. (2004), Briec et al. (2007). Section 3 develops all geometrical representations and their interpretations in a series of subsections. Based mainly on the GP literature in OR, it also clarifies some ambiguities one can find in current portfolio applications of the PGP approach. First, PGP results are

[1] Aboul-Enein et al. (2012), Anson (2006), Elkaim and Papageorgiou (2006) all develop a four moment PGP model as well. Hafner and Wallmeier (2008) also normalize the deviation variables, but their portfolio model only has implicitly four dimensions: they maximize skewness and minimize kurtosis, but maximize the Sharpe ratio to combine the first two moments.

[2] For an expected utility optimizer, the more basic equivalence between multi-moment preferences and a polynomial utility function of corresponding degree is proven in, e.g., Müller and Machina (1987).

[3] We are unaware of other systematic contributions on higher order moment reconstruction. Obviously, geometric reconstructions play a major role in practical portfolio management. E.g., Anagnostopoulos and Mamanis (2010) extend the MV space with one dimension representing the optimal number of assets.

interpreted within the context of a MVS reconstruction based on the shortage function following Kerstens et al. (2011). Then, it is reminded that PGP results are MVS efficient. Next, we explore the ways of developing geometrical representations using the PGP approach. Finally, we discuss some geometrical representations of portfolio models PGP currently fails to generate while the shortage function approach manages to deliver. To remedy these problems, we propose a new, revised PGP formulation allowing for two-dimensional reconstructions in more general portfolio settings. The resulting reconstructions are similar to the new, two-dimensional reconstruction formulations based on the shortage function introduced earlier. Section 4 presents two new results to the literature. First, we establish a formal relation between the shortage function and the PGP approach. Second, following the geometrical representations based on the PGP and shortage function approaches, we offer a substantial result showing that a single mean-skewness (MS) section is sufficient to reconstruct the MVS frontier under a risk-free asset with shorting. This is a generalized one fund separation result. Section 5 concludes the paper.

2. Basic portfolio framework and methodologies

We start by introducing the basic portfolio setting and the two methodologies for determining optimal MVS portfolios: the PGP approach, and the shortage function method.

2.1. Basic portfolio framework

Consider the problem of selecting a portfolio from n financial products (addressed as assets hereafter, though other risky products could be considered) and a risk-free asset.

A portfolio $(w, w_{Rf}) = (w_1, \ldots, w_n, w_{Rf}) \in \mathbb{R}^{n+1}$ is a vector of proportions in each of these n financial assets and the risk-free asset with $\sum_{i=1}^{n} w_i + w_{Rf} = 1$. If a risk-free asset is absent or cannot be selected, then $w_{Rf} = 0$. If shorting is not allowed, then all proportions w_i, for $i \in \{1, \ldots, n\}$, and w_{Rf} must be positive. The set of admissible portfolios is denoted by $\Im$.

The assets from which the investor makes a choice are characterized by their returns R_i, for $i \in \{1, \ldots, n\}$. From these, the expected return vector $E[R]$ can be derived, as also the covariance matrix Ω, and the coskewness tensor of rank three Λ. Obviously, the risk-free asset has expected return R_{Rf} and zero covariances and coskewnesses with itself and the other assets.

The return of portfolio (w, w_{Rf}) is defined by $R(w, w_{Rf}) = \sum_{i=1}^{n} w_i R_i + w_{Rf} R_{Rf}$. The expected return of portfolio (w, w_{Rf}), its variance and skewness are computed as follows:

$$E[R(w, w_{Rf})] = \sum_{i=1}^{n} w_i E[R_i] + w_{Rf} R_{Rf}, \tag{1}$$

$$Var[R(w, w_{Rf})] = E[(R(w, w_{Rf}) - E[R(w, w_{Rf})])^2] = \sum_{i,j=1}^{n} w_i w_j \Omega_{ij}, \tag{2}$$

$$Sk[R(w, w_{Rf})] = E[(R(w, w_{Rf}) - E[R(w, w_{Rf})])^3] = \sum_{i,j,k=1}^{n} w_i w_j w_k \Lambda_{ijk}. \tag{3}$$

Note that skewness refers to the third central moment in the remainder of this text.

Let $\Phi : \Im \to \mathbb{R}^3$ be the function defined by

$$\begin{aligned}\Phi(w, w_{Rf}) &= (\Phi_M(w, w_{Rf}), \Phi_V(w, w_{Rf}), \Phi_S(w, w_{Rf})) \\ &= (E[R(w, w_{Rf})], Var[R(w, w_{Rf})], Sk[R(w, w_{Rf})]).\end{aligned}$$

It provides the expected return, variance and skewness of a given portfolio (w, w_{Rf}). The functions Φ_M, Φ_V and Φ_S represent the coordinate functions of Φ.

In the remainder, an arbitrary element $\alpha = (\alpha_M, \alpha_V, \alpha_S)$ of $\mathbb{R}^3$ is called a MVS point. Thus, a MVS point can be the image by Φ of a portfolio, or any arbitrary point in this three-dimensional space. The MVS image of $\Im$ is obtained by $\Phi(\Im) = \{\Phi(w, w_{Rf}); (w, w_{Rf}) \in \Im\}$. This set can be extended by defining a MVS disposal representation set $\mathcal{DR} = \Phi(\Im) + (\mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_-)$.

Instead of working with expected return, variance and skewness, it is sometimes convenient to switch to the normalized moments determined by expected return, normalized variance (i.e., square root of variance or standard deviation) and normalized skewness (i.e., cubic root of skewness). Using the letter 'n' in the notation when referring to normalized moments (e.g., *nVar* refers to normalized variance), we also introduce the normalized function $n\Phi : \Im \to \mathbb{R}^3$ mapping an arbitrary portfolio into normalized MVS space:

$$\begin{aligned}n\Phi(w, w_{Rf}) &= (E[R(w, w_{Rf})], nVar[R(w, w_{Rf})], nSk[R(w, w_{Rf})]) \\ &= \left(E[R(w, w_{Rf})], \sqrt{Var[R(w, w_{Rf})]}, \sqrt[3]{Sk[R(w, w_{Rf})]}\right).\end{aligned}$$

2.2. PGP model

In this section, we introduce notation related to the multi-objective approach for selecting MVS optimal portfolios in Lai (1991). His main idea in developing a feasible MVS model assuming shorting and a risk-free asset is to search for a portfolio maximizing both expected excess return Z_1 and skewness Z_3 for a given level of variance Z_2. Obviously, $Z_1 = E[R(w, w_{Rf})] - R_{Rf}$. Under the assumption of shorting, the expected excess return Z_1 can become arbitrary large making the maximization of Z_1 unbounded. Therefore, some restriction is needed: Lai (1991) focuses on unit variance portfolios solely. Under the current assumptions, this makes sense since a non-unit variance portfolio can always be rescaled. This guarantees that the problem of maximizing both expected excess return Z_1 and skewness Z_3 is feasible.

Ideally, one searches for a unit variance portfolio maximizing both Z_1 and Z_3. However, since it is unlikely (if not impossible) to achieve both goals simultaneously, the PGP program of Lai (1991) and adopted in the ensuing literature finds a compromise:

Definition 2.1. For given parameter values $\alpha, \beta \in \mathbb{R}_+$, the *PGP model* is defined by

$$PGP(\alpha, \beta) = \min_{(w, w_{Rf}) \in \Im} \left\{ d_1^{\alpha} + d_3^{\beta}; d_1 = Z_1^* - Z_1, d_3 = Z_3^* - Z_3, Z_2 = 1 \right\},$$

with

$$Z_1^* = \max_{(w, w_{Rf}) \in \Im} \{Z_1; Z_2 = 1\} \tag{4}$$

and

$$Z_3^* = \max_{(w, w_{Rf}) \in \Im} \{Z_3; Z_2 = 1\}. \tag{5}$$

Thus, the target expected excess return Z_1^* and the target portfolio skewness Z_3^* are first determined in two separate portfolio optimization programs: (i) The program determined by (4) maximizes expected excess return subject to a unit portfolio variance constraint; (ii) The program determined by (5) maximizes portfolio skewness subject to the same unit portfolio variance constraint. The $PGP(\alpha, \beta)$ program simultaneously minimizes the deviations between expected excess return and its target, and between portfolio skewness and its target, subject to the unit portfolio variance constraint. The powers α and β in Definition 2.1 related to the deviation variables d_1 and d_3 are determined according to the investor's preferences towards expected excess return and skewness. A larger value of α reflects a higher importance of maximizing expected excess return, while a larger value of β corresponds with a higher interest in maximizing portfolio skewness.

In the remainder, we focus on the initial Lai (1991) formulation and ignore any variations on this basic framework. For instance, Davies et al. (2009) normalize the deviation variables and have a slightly different objective function.[4]

2.3. Shortage function model

Following Kerstens et al. (2011), we now adapt the shortage function to the basic assumptions of having a risk-free asset and short selling. Moreover, we include additional adapted versions suitable for generating a geometrical representation of the MS frontier, among others. We start with the shortage function in MVS space introduced in the following definition:

Definition 2.2. Let $g=(g_M,g_V,g_S)\in\mathbb{R}_+\times\mathbb{R}_-\times\mathbb{R}_+$ and $g\neq 0$. The *shortage function* $\mathcal{S}_g$ in the direction of vector g is the real valued function defined by $\mathcal{S}_g(y)=\sup_{\delta\in\mathbb{R}}\{\delta; y+\delta g\in\mathcal{DR}\}$.

Starting from a given MVS point, the shortage function seeks to simultaneously improve expected return and skewness and reduce variance in the direction of vector g. The choice of this direction vector depends on investor's preferences. Note that the shortage function is an efficiency gauge whereby a zero value indicates efficiency.

The computation of the shortage function value specified in Definition 2.2 for a MVS point $y=(y_M, y_V, y_S)$ in the direction of the vector $g=(g_M, g_V, g_S)$ is obtained by solving the following cubic non-linear programming model:

$$\mathcal{S}_g(y)=\max_{(w,w_{Rf})\in\mathfrak{I}}\{\delta; E[R(w,w_{Rf})]\geqslant y_M+\delta g_M, Var[R(w,w_{Rf})]\leqslant y_V+\delta g_V, Sk[R(w,w_{Rf})]\geqslant y_S+\delta g_S\}. \tag{6}$$

When solving model (6), the efficiency value $\mathcal{S}_g(y)$ as well as the left-hand and right-hand sides of the constraints in the optimal value provide useful information. Denoting the optimal value by δ^* and the optimal portfolio by $\left(w^*,w^*_{Rf}\right)$, then the MVS point derived from the left-hand sided of the constraints $\left(E\left[R\left(w^*,w^*_{Rf}\right)\right],Var\left[R\left(w^*,w^*_{Rf}\right)\right],Sk\left[R\left(w^*,w^*_{Rf}\right)\right]\right)$ is always positioned on the strongly efficient frontier (see Kerstens et al., 2011). The MVS point deducted from the right-hand sides of the constraints $(y_M+\delta^*g_M,\ y_V+\delta^*g_V,\ y_S+\delta^*g_S)$ is always situated on the weakly efficient frontier. Thus, $\mathcal{S}_g(y)=0$ if and only if the MVS point y is positioned on the weakly efficient frontier.

One can force possible slacks between left-and right-hand sides of the constraints in (6) to be zero by replacing the inequalities with equalities:

$$\mathcal{S}^{=}_g(y)=\max_{(w,w_{Rf})\in\mathfrak{I}}\{\delta; E[R(w,w_{Rf})]=y_M+\delta g_M, Var[R(w,w_{Rf})]=y_V+\delta g_V, Sk[R(w,w_{Rf})]=y_S+\delta g_S\}. \tag{7}$$

If an optimal solution is found, then the corresponding MVS point obtained from the left-hand sides of the equality constraints is positioned on the boundary of $\Phi(\mathfrak{I})$.

One of the variables can even be dropped to obtain two-dimensional shortage functions relative to a more basic two-dimensional portfolio model. For instance, if the skewness is omitted in (6), then the MV shortage function is obtained:

$$\mathcal{S}^{-S}_g(y)=\max_{(w,w_{Rf})\in\mathfrak{I}}\{\delta; E[R(w,w_{Rf})]\geqslant y_M+\delta g_M, Var[R(w,w_{Rf})]\leqslant y_V+\delta g_V\}. \tag{8}$$

[4] See also Canela and Collazo (2007), among others.

3. Revisiting the Lai (1991) contribution in view of the OR literature

To illustrate the PGP framework set out in Section 2.2, we use the example from Lai (1991) for which all statistics are publicly available. In particular, the risk-free return, expected returns, covariance matrix and coskewness tensor are published in Lai (1991). Indeed, we simply take the risk-free asset of 0.0058 and the expected returns from Table 1 on page 298 in Lai (1991). The covariance matrix and coskewness tensor are available from the table in the appendix on page 302 in the same article.

This section mainly focuses on a geometric interpretation of the Lai (1991) contribution and the clarification of some results known in the GP literature in OR (see mainly Miettinen, 1999, Steuer, 1986). Seemingly, these results are not necessarily known in the literature applying the Lai (1991) model in finance. Thus, the aim is to clarify some ambiguities in this PGP portfolio literature initiated by Lai (1991).[5]

3.1. PGP results within the context of a MVS reconstruction based on the shortage function

Since one can obtain points on the weakly and strongly efficient frontier and on the boundary of $\Phi(\mathfrak{I})$ by applying the shortage functions $\mathcal{S}_g$ and $\mathcal{S}^{=}_g$ (see (6) and (7)), one can come up with techniques for generating portfolio frontiers. Summarizing Kerstens et al. (2011), it turns out that projecting some planar grid in fixed directions parallel to the coordinate axes obtains the best results.

This technique of projecting planar grids in fixed directions parallel to the coordinate axes generates Fig. 1.[6] More precisely, the shortage functions (6) and (7) are used with direction vector $g=(0, 0, 1)$ starting from a single planar grid situated in the MV plane. Fig. 1a represents the frontier in MVS space, while Fig. 1b shows the frontier in the space of mean return, normalized variance (i.e., standard deviation) and normalized skewness (i.e., cubic root of skewness). Part of the MVS frontier is visualized in Fig. 1 as a point cloud. This MVS frontier consists of the upper part of the boundary of $\Phi(\mathfrak{I})$ when looking in the skewness direction (i.e., the vertical direction in the figure). The horizontal plane is the MV plane.

The MV frontier visible in Fig. 1 is computed by means of the MV shortage function (8). Note that the skewness of all these MV optimal portfolios is used for visualizing this MV frontier, embedding the observed MV frontier into MVS space. Clearly, it is the lower boundary of the MVS frontier. Projecting this embedded MV frontier into the MV plane yields a classical two-dimensional MV curve.

Furthermore, observe in Fig. 1 the vertical unit variance plane intersecting the MVS frontier along a planar curve. This curve is referred to as MS frontier. A two dimensional visualization is presented in Figure B.2 in Appendix B.

This MS section can also be generated by the shortage function (7) with the value of the portfolio variance fixed at unity ($V_0=1$). This leads to the introduction of the notion of a *variance fixed shortage function.*

Definition 3.1. Let $g=(g_M,g_V,g_S)\in\mathbb{R}_+\times\mathbb{R}_-\times\mathbb{R}_+$ and $g\neq 0$. The *variance fixed shortage function* $\mathcal{S}^{V=V_0}_g$ in the direction of vector g and fixed at variance level $V=V_0$ is the function $\mathcal{S}^{V=V_0}_g:\mathbb{R}^3\to\mathbb{R}\cup\{-\infty,+\infty\}$, with

$$\mathcal{S}^{V=V_0}_g(y)=\max_{(w,w_{Rf})\in\mathfrak{I}}\{\delta; E[R(w,w_{Rf})]\geqslant y_M+\delta g_M, Var[R(w,w_{Rf})]=V_0, Sk[R(w,w_{Rf})]\geqslant y_S+\delta g_S\}. \tag{9}$$

[5] Note that Gan (2001) aims to point out some other pitfalls in the same literature.
[6] All figures in this contribution are generated using Maple version 14.

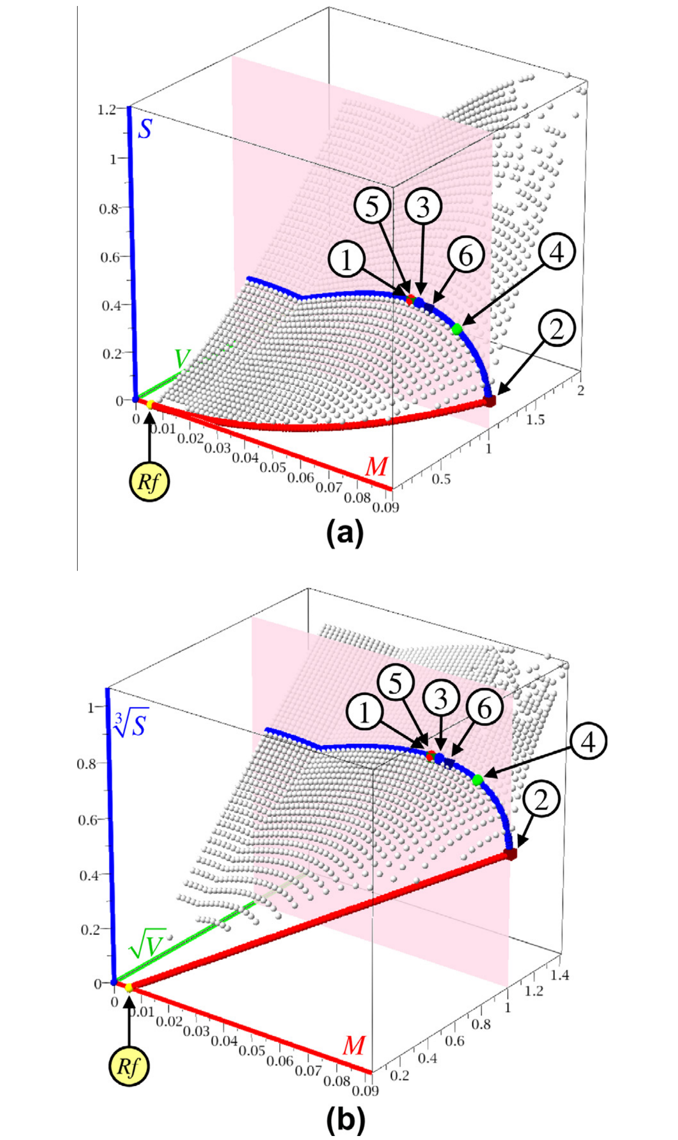

Fig. 1. Geometrical representation of the (a) MVS frontier and (b) normalized MVS frontier, the intersection with the unit variance plane and the position of some PGP optimal portfolios.

This new, special formulation is crucial to establish a link with the PGP approach. Note that the portfolio variance is now fixed (V_0) instead of being linked to some portfolio under evaluation (as in (6) and (7)). Also note that the variance component g_V of the direction vector g is put to zero: it has no influence on the variance fixed shortage function value. Thus, the direction vector becomes (g_M, 0, g_S). In particular, this variance fixed shortage function allows reconstructing the MS frontier from a line grid along the fixed variance dimension and covering the range of the return dimension by projection into the remaining skewness dimension (i.e., g = (0, 0, 1)).[7]

The PGP points computed with the powers indicated in the headings of Tables 2 and 3 on pages 299–300 in Lai (1991) are shown by the labeled points in Fig. 1 and Figure B.2 in Appendix B. Data on these same six PGP points in Fig. 1 is found in Table 1.

[7] Alternatively, one can obtain the MS frontier from a grid along the range of the skewness dimension by projection along the return dimension for a given fixed variance using the direction vector (1, 0, 0).

In addition to the first two parts containing the labels in all figures and the powers for α and β from Lai (1991), Table 1 contains six parts separated by horizontal lines. The first part contains the optimal values of the PGP objective function. Then, we have the optimal values d_1^* and d_3^* of the deviation variables d_1 and d_3 respectively. The third part reports the optimal portfolio weights $\left(w^*, w^*_{Rf}\right)$. Part four shows the MVS coordinates of the optimal PGP point (obviously, its variance equals one). Finally, the last two parts list the values obtained from applying the MVS ($\mathcal{S}_g$) and MV $\left(\mathcal{S}_g^{-S}\right)$ shortage functions on the optimal PGP points in part four.

Currently, we have three main comments on Table 1. First, the optimal value of the PGP objective function has no straightforward interpretation in terms of investor preferences (see also Gan (2001) for more details and an illustration (i.e., his Table 1)). For α = 0 and β = 1, for instance, the optimal deviation values $d_1^* = 0.027$ and $d_3^* = 0$ indicate a positive deviation from the target expected excess return and no deviation from the target portfolio skewness, respectively. This exclusive preference for skewness yields an objective function value of unity. Exactly the opposite preferences (i.e., α = 1 and β = 0) yield also an objective function value of unity, even though now there is no deviation from the target expected excess return and a larger positive deviation from the target portfolio skewness (i.e., $d_1^* = 0$ and $d_3^* = 0.309$). Thus, different deviation variables and opposite powers yield an identical objective function value of unity. Second, observe that the variance of all optimal solutions equals unity. Third, we are unable to duplicate the original results in Lai (1991) (compare to his Tables 2 and 3 on pages 299–300).[8]

3.2. PGP results are MVS efficient: a remark

This section establishes the efficiency status of the PGP optimal points by reference to the existing goal programming literature. Sometimes doubts on the efficiency status of the PGP approach are expressed. For instance, Jurczenko et al. (2006) conjecture in a MVS kurtosis context that "minimizing deviations from the first four moments simultaneously only guarantees a solution close to the mean–variance–skewness–kurtosis efficient frontier." (page 52). By contrast, Chunhachinda et al. (1997) claim that the "existence of an optimal solution" is a key feature of the PGP approach (page 146). The MVS efficiency of PGP is now formally established in a remark contradicting the above conjecture.

Proposition 3.1. *For any l_p norm, all PGP optimal portfolios are MVS efficient.*

The proof for this efficiency follows from the GP literature: e.g., Miettinen (1999, Theorem 2.1.1, pp. 67–70). Note that in the case of the shortage function the MVS efficiency is also well established. In general, Briec and Kerstens (2010) prove that the shortage

[8] This is due to several reasons. First, we do not have the original raw returns from which the statistical data has been computed. The unavoidable rounding of these derived statistics in an article leads to rather drastic changes in optimal solutions. In fact, Lai (1991) reports the expected returns at 3 decimals, the variance–covariance at 4 decimals, and the skewness–coskewness tensor at 5 decimals. Second, the optimization is non-linear in nature. Therefore, optimization algorithms may end up in local rather than global optima. Neither the numerical optimization routine used in Lai (1991) (see his footnotes 9, 16, 17 and 19), nor the one applied here guarantee that global optima are obtained. However, we apply a pseudo-global optimizer by repeating the local optimization process multiple times from different initial positions, thereby increasing the probability of reaching a global optimum. Third, numerical optimization routines as well as hardware have substantially improved over these two decades leading to more accurate results. To be consistent, we continue to reason with the solutions obtained rather than copying the original results in Lai (1991).

Table 1
The results from optimizing with PGP for distinct combinations of α and β and the projection of these optimal portfolios using the MVS and MV shortage functions.

Label	①	②	③	④	⑤	⑥
Combination of α and β	$\alpha = 0, \beta = 1$	$\alpha = 1, \beta = 0$	$\alpha = 1, \beta = 1$	$\alpha = 1, \beta = 2$	$\alpha = 2, \beta = 1$	$\alpha = 2, \beta = 2$
Optimal *PGP* value	1.000000	1.000000	0.025992	0.014725	0.000747	0.000499
d_1^*	0.027419	0.000000	0.024624	0.011338	0.027259	0.020848
d_3^*	0.000000	0.309610	0.001368	0.058193	0.000004	0.008050
w_1^*	−0.482121	1.005840	−0.381492	0.155816	−0.476493	−0.240156
w_2^*	−0.016725	−0.566768	−0.097031	−0.454698	−0.021583	−0.202516
w_3^*	1.839502	1.450752	1.844821	1.821488	1.839928	1.847499
w_4^*	−1.288837	0.897412	−1.162049	−0.428958	−1.281658	−0.978824
w_5^*	4.485886	2.067465	4.429692	3.909411	4.483054	4.329763
w_{Rf}^*	−3.537705	−3.854701	−3.633940	−4.003059	−3.543249	−3.755766
$E\left[R\left(w^*, w_{Rf}^*\right)\right]$	0.064803	0.092222	0.067598	0.080883	0.064962	0.071374
$Var\left[R\left(w^*, w_{Rf}^*\right)\right]$	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
$Sk\left[R\left(w^*, w_{Rf}^*\right)\right]$	0.428260	0.118650	0.426893	0.370067	0.428256	0.420210
Optimal value $\mathcal{S}_g\left(\Phi\left(w^*, w_{Rf}^*\right)\right)$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
Optimal value $\mathcal{S}_g^{-S}\left(\Phi\left(w^*, w_{Rf}^*\right)\right)$	0.246856	0.000000	0.217051	0.090619	0.245126	0.178633

function guarantees a global optimal solution for a large class of convex problems. Furthermore, it yields a set of weakly efficient portfolio solutions that contains at least one strongly efficient solution.[9] In particular, Briec et al. (2007) establish sufficient conditions to guarantee MVS efficiency. These results clearly make the shortage function and PGP approaches stand out compared to some of the other primal approaches listed in the introduction where efficiency claims seem only rarely established.

Observe in Fig. 1 that all MVS points are at the intersection of the MVS frontier with the unit variance vertical plane. In Table 1, the PGP optimal unit variance portfolios are computed for distinct values of parameters α and β. Computing both the MVS (8) and the MV (6) shortage functions using the position dependent direction vector for all these points, optimal values are reported in the last two rows of Table 1. Since all shortage function values equal zero, this illustrates that all PGP optimal portfolios are MVS efficient. Focusing on the last row of Table 1 computed using the MV shortage function (8), this function equals zero (i.e., MV efficient) only for the PGP point obtained with $\alpha = 1$ and $\beta = 0$ (these parameter values attach no importance to skewness). All other combinations of α and β lead to MV inefficient points.

3.3. Geometrical interpretation of PGP and its representation

As observed in Sections 3.1 and 3.2, varying the values of the parameters α and β in Definition 2.1 leads to different MVS efficient points on the MS frontier. Therefore, these points are also part of the MVS frontier. By altering the values of the parameters α and β, different regions on the MS frontier can be reached.

Gan (2001), Zghal et al. (2011) remark that the PGP problem involves a relation between an exterior ideal point constituted by the optimal solutions to the target expected excess return Z_1^* and the target portfolio skewness Z_3^* on the one hand, and the PGP optimal points on the MVS frontier on the other hand. An illustration of this ideal point can be found in Figure B.2 in Appendix B. This observation leads to the following questions: (i) What is the exact geometrical relation between this exterior ideal point and the PGP optimal points? This calls for a geometric interpretation of the PGP optimization framework. (ii) Can one use the PGP model for the geometric reconstruction of portfolio frontiers? Remark that while the geometry of GP models is well-known in general terms for l_p norms (see, e.g., Miettinen, 1999, p. 69), it has to our knowledge been barely developed more generally and within the PGP portfolio context following Lai (1991).[10]

Starting with the first question, we must more clearly understand how PGP optimization determined by Definition 2.1 geometrically selects points on the MS frontier. Put differently, we need to grasp the relation between the choice of the parameters α and β and the resulting position on the MS frontier. From Definition 2.1, it follows that

$$PGP(\alpha, \beta) = \min_{(w, w_{Rf}) \in \Im} \left\{ \left|Z_1 - Z_1^*\right|^{\alpha} + \left|Z_3 - Z_3^*\right|^{\beta}; Z_2 = 1 \right\}.$$

Denote the expected return of the optimal portfolio obtained from problem (4) by P_M and the skewness of the portfolio solving problem (5) by P_S. Then, $Z_1 - Z_1^* = \Phi_M(w, w_{Rf}) - P_M$ and $Z_3 - Z_3^* = \Phi_S(w, w_{Rf}) - P_S$. Consequently,

$$PGP(\alpha, \beta) = \min_{(w, w_{Rf}) \in \Im} \{\bar{r}; \bar{r} = |\Phi_M(w, w_{Rf}) - P_M|^{\alpha} + |\Phi_S(w, w_{Rf}) - P_S|^{\beta} \text{ and } \Phi_V(w, w_{Rf}) = 1\}. \tag{10}$$

Thus, when solving (10), one needs to look for unit variance portfolios (w, w_{Rf}) minimizing

$$\bar{r} = |\Phi_M(w, w_{Rf}) - P_M|^{\alpha} + |\Phi_S(w, w_{Rf}) - P_S|^{\beta}.$$

Since the point $(\Phi_M(w, w_{Rf}), \Phi_S(w, w_{Rf}))$ is contained in the unit variance MS plane, it is natural to consider the geometrical object in this MS plane described by the implicit equation

$$|M - P_M|^{\alpha} + |S - P_S|^{\beta} = \bar{r}, \tag{11}$$

for some fixed $\bar{r} \in \mathbb{R}_+$.

Consider the example of parameter values $\alpha = 2$ and $\beta = 2$. Then, (11) simplifies to $(M - P_M)^2 + (S - P_S)^2 = \bar{r}$, which represents a circle with radius $\sqrt{\bar{r}}$ and center (P_M, P_S).

Since we prefer not to have the square root in the previous special case, we propose the transformation $\bar{r} = r^{\frac{1}{2}(\alpha+\beta)}$ for the general case. Consequently, (10) is rewritten as

[9] Hence, if a unique solution exists, then it is strongly efficient.

[10] Note that Gan (2001) develops part of our analysis in a somewhat more general portfolio setting.

$$PGP(\alpha,\beta) = \left(\min_{(w,w_{Rf})\in\Im}\left\{r; r^{\frac{1}{2}(\alpha+\beta)} = |\Phi_M(w,w_{Rf}) - P_M|^{\alpha} + |\Phi_S(w,w_{Rf}) - P_S|^{\beta} \text{ and } \Phi_V(w,w_{Rf}) = 1\right\}\right)^{\frac{1}{2}(\alpha+\beta)}, \quad (12)$$

and (11) becomes

$$|M - P_M|^{\alpha} + |S - P_S|^{\beta} = r^{\frac{1}{2}(\alpha+\beta)}. \quad (13)$$

It is easily verified that the geometrical object described by (13) is the circle in the unit variance MS plane with radius r and center (P_M, P_S) in the case $\alpha = 2$ and $\beta = 2$. Therefore, we propose the following definition:

Definition 3.2. The *PGP circle* in the unit variance MS plane with radius r and center (P_M, P_S) for parameter values α and β is the geometrical object described by the implicit equation

$$|M - P_M|^{\alpha} + |S - P_S|^{\beta} = r^{\frac{1}{2}(\alpha+\beta)}.$$

Clearly, the notion of a 'circle' should be relaxed. Only for the special case of $\alpha = 2$ and $\beta = 2$, we obtain regular Euclidean circles. We refer to Section B.2 in Appendix B for more information and illustrations.

In the general case, a smaller r results in a smaller PGP circle. Consequently, (12) determines in the unit variance MS plane the minimal possible radius r of a PGP circle around (P_M, P_S) that remains in contact with $\Phi(\Im)$. Clearly, the PGP circle is tangent to the boundary of $\Phi(\Im)$ at the minimum.

Proposition 3.2. *The PGP circle in the unit variance MS plane with radius $PGP(\alpha,\beta)^{\frac{2}{\alpha+\beta}}$ and center (P_M, P_S) for parameter values α and β, and the unit variance MS section of the MVS frontier are locally tangent to each other at the MVS image of the PGP optimal portfolio.*[11]

This proposition is illustrated in Fig. 2 for the same parameter settings as in Figure B.3 of Appendix B. Additional figures are again available this same appendix. Note that the proportions of the PGP circles in Fig. 2 differ from those in Figure B.3 because of a different scaling of the axes.

Note that the point (P_M, P_S) derived from the two-dimensional problems (4) and (5) offers an ultimate, but unreachable goal since it is situated outside the MS section of the portfolio frontier. We label this point 'Ideal portfolio with maximal return and maximal skewness'. Around this point, the radius minimizing PGP circle is drawn: a PGP circle is tangent to the boundary of $\Phi(\Im)$ in a point visualized by a small square (□) in the figure. This tangency point is the MS point of the portfolio minimizing the radius.

Turning to the second question, we now highlight four potential difficulties. First, understanding the geometry behind PGP, we now return to the idea of varying the values of α and β to compute a sufficient number of points on the MS frontier for it to become visible. We know that this variation leads to differently shaped PGP circles, each tangent to the boundary of $\Phi(\Im)$. Compared to the shortage function approach where an initial point is projected according to a direction vector, the link between the parameter values for α and β and the PGP circle on the one hand and the resulting boundary point on the other hand is more involved. This directly leads to a first difficulty of controlling the position of this optimal PGP point.

A second difficulty is related to the presence of two degrees of freedom for generating the MS frontier which is a curve (i.e., a one-dimensional object). In general, increasing the value of α increases the importance of maximizing expected return, shifting the optimal PGP point down along the MS frontier. Similarly, increasing the value of β increases the weight of maximizing skewness, resulting in an upward shift along the MS frontier. However, by increasing both α and β, effects from the first parameter partially outweight those from the second parameter. Thus, identical or neighboring PGP optimal points can be found with quite different parameter settings.

Third, the parameters α and β can vary over an infinitely large interval, i.e., the set $\mathbb{R}_+$. Therefore, some extreme regions are probably hard to reach. This implies that one should make a correct selection in the combination of parameter values. This problem can be remedied by a transformation mapping the infinitely large parameter domain to a finite interval, picking appropriate values in this finite domain and then mapping these back into the original set. Quite a few transformations performing these operations are available. One example is the function $f(x) = \frac{x}{1+x}$ with inverse $f^{-1}(y) = \frac{y}{1-y}$: f maps the infinitely large interval $[0, +\infty)$ into the finite length interval $[0, 1)$.

Despite these reservations, it turns out that the PGP model determined by Definition 2.1 manages to rather decently reconstruct the efficient subset of the unit variance MS section. As anticipated, the distribution of points can be more or less even depending on the choice of the two parameters α and β.[12] Since it is rather well-known that GP models based on l_p norms with $p < \infty$ do not guarantee that all strongly efficient solutions are found, while GP formulations based on the l_∞ norm (which is similar to the shortage function-see Proposition 4.1 below) do manage to find all strongly efficient solutions (see, e.g., Šipošová, 2008), the issue of the quality of PGP based reconstructions remains to be further explored.[13]

3.4. Extending PGP to alternative portfolio models

In this section, we turn to the question whether PGP can handle other portfolio models in its current formulation. Lai (1991) claims that the assumption of short selling is non-essential (see footnote 6 on page 303). PGP results without shorting are reported in Chang et al. (2008a) and in Prakash et al. (2003), among others. Thus, it seems the PGP approach can also impose non-negativity on all portfolio weights, excluding short selling.

However, this claim is incorrect. In Appendix B, this is illustrated with two figures: Figure B.10 without shorting and without risk-free asset, and Figure B.11 with a risk-free asset but without shorting. It is now possible to observe that the maximal value for the optimal portfolio variances along the variance axis or the standard deviation axis is way below the unit level, making the PGP model based on a unit variance constraint infeasible. Thus, the claim of Lai (1991) is unfounded in general.

But, in these cases it may suffice to use other values for the variance constraint to guarantee feasibility of the optimization process proposed in Lai (1991). This implies generalizing the definition of the PGP model described by Definition 2.1 to this new one:

Definition 3.3. For given parameter values $\alpha, \beta \in \mathbb{R}_+$ and for some variance level V_0, the *generalized PGP model* is defined by

$$PGP^{V_0}(\alpha,\beta) = \min_{(w,w_{Rf})\in\Im}\left\{d_1^{\alpha} + d_3^{\beta}; d_1 = Z_1^*(V_0) - Z_1, d_3 = Z_3^*(V_0) - Z_3, Z_2 = V_0\right\},$$

with

[11] Note that all proofs of propositions are in Appendix A.

[12] See Figure B.10 in Section B.4 in Appendix B.

[13] Šipošová (2008) even proves that under certain conditions the l_p norms (with $p < \infty$) as well as the l_∞ norm can determine strongly efficient solutions.

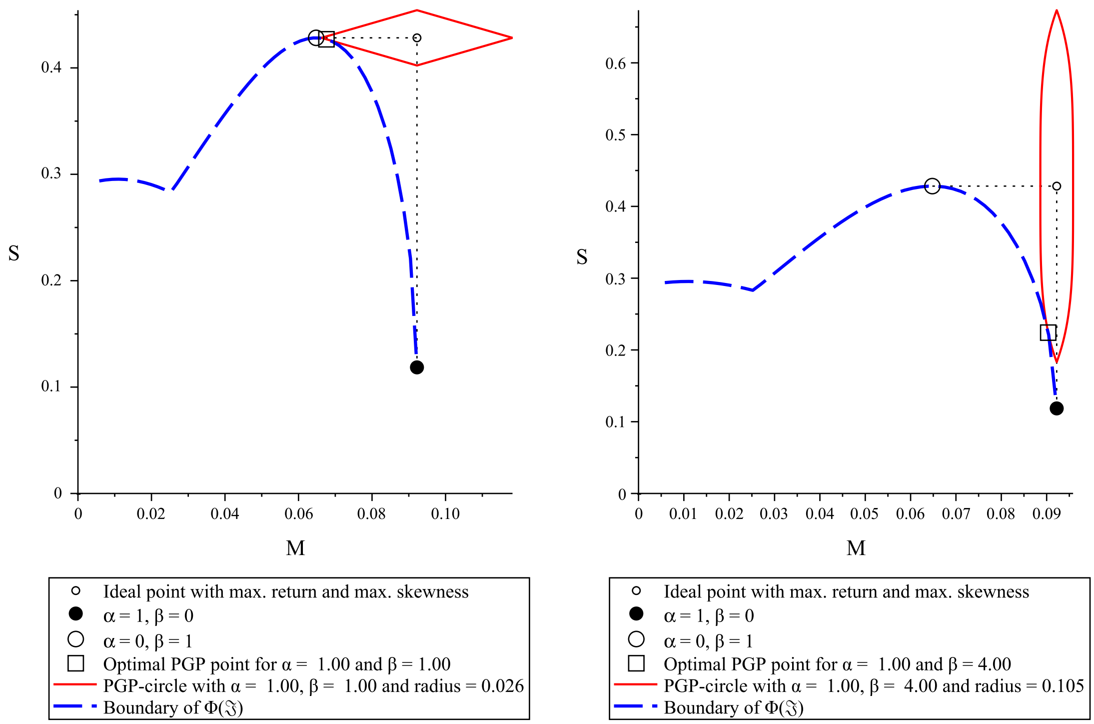

Fig. 2. Visualization of the PGP optimization process for different values of α and β.

$$Z_1^*(V_0) = \max_{(w,w_{Rf})\in\Im}\{Z_1; Z_2 = V_0\} \tag{14}$$

and

$$Z_3^*(V_0) = \max_{(w,w_{Rf})\in\Im}\{Z_3; Z_2 = V_0\}. \tag{15}$$

While it may be a priori difficult to know which constraining values should be imposed to make the PGP model feasible in such contexts, it is straightforward to come up with a workable empirical strategy. One simple solution is to fix a variance level within the range of variance levels observed in the underlying return data. This guarantees that the PGP approach is feasible.

Proposition 3.3. *The generalized PGP model is feasible if V_0 is situated between the minimal and maximal possible variance levels observed in the underlying return data.*

Consider some target excess return Z_1^t, variance Z_2^t and skewness Z_3^t. Furthermore, denote the absolute differences $d_i = |Z_i - Z_i^t|$ with $i \in \{1,2,3\}$. Then, the following generalized three dimensional PGP model can be considered:

Definition 3.4. For given parameter values $\alpha, \beta, \gamma \in \mathbb{R}_+$ and a subset $\mathcal{A} \subset \Im$, the *generalized three dimensional PGP model* is defined by

$$PGP_{\mathcal{A}}(\alpha, \beta, \gamma; Z_1^t, Z_2^t, Z_3^t) = \min_{(w,w_{Rf})\in\mathcal{A}}\left\{d_1^{\alpha} + d_2^{\gamma} + d_3^{\beta}\right\}.$$

Obviously, when $Z_1^t = Z_1^*(V_0)$, $Z_3^t = Z_3^*(V_0)$, $Z_2^t = V_0$ and $\mathcal{A} = \{(w, w_{Rf}) \in \Im; Z_2 = V_0\}$, then $PGP_{\mathcal{A}}(\alpha, \beta, 1; Z_1^t, Z_2^t, Z_3^t) = PGP^{V_0}(\alpha, \beta)$. This generalized three dimensional PGP model in Definition 3.4 has a structure similar to Leung et al. (2001, formulation (26)) and remedies the pitfalls described in Gan (2001) common to a variety of models based on Lai (1991).

4. Comparing PGP and shortage function: new results

Our systematic comparison of PGP and shortage function approaches leads to two theoretical results that are new to the literature. The second finding is again illustrated by geometric representations in MVS space based on Lai (1991) data.

4.1. PGP and shortage function: a relation

From Definition 3.3, we have

$$PGP^{V_0}(\alpha, \alpha) = \min_{(w,w_{Rf})\in\Im}\left\{\left|Z_1 - Z_1^*(V_0)\right|^{\alpha} + \left|Z_3 - Z_3^*(V_0)\right|^{\alpha}; Z_2 = V_0\right\}.$$

For all $\alpha > 0$, it follows that

$$[PGP(\alpha, \alpha)]^{\frac{1}{\alpha}} = \min_{(w,w_{Rf})\in\Im}\left\{\left(\left|Z_1 - Z_1^*(V_0)\right|^{\alpha} + \left|Z_3 - Z_3^*(V_0)\right|^{\alpha}\right)^{\frac{1}{\alpha}}; Z_2 = V_0\right\}. \tag{16}$$

Note that this PGP formulation (16) has a similar structure to the formulation in Leung et al. (2001).

In addition, we denote

$$PGP_{\infty}^{V_0} = \min_{(w,w_{Rf})\in\Im}\left\{\max\left\{\left|Z_1 - Z_1^*(V_0)\right|, \left|Z_3 - Z_3^*(V_0)\right|\right\}; Z_2 = V_0\right\}.$$

Proposition 4.1. *Consider the vectors $\mathbb{1}_2 = (1,1)$ and $Z^* = (Z_1^*(V_0), V_0, Z_3^*(V_0))$. Assume there exists some $\delta \in \mathbb{R}_-$ and some $(w, w_{Rf}) \in \Im$ such that $Z_1 \geqslant Z_1^*(V_0) + \delta, Z_2 = V_0$ and $Z_3 \geqslant Z_3^*(V_0) + \delta$. Then, $\mathcal{S}_{\mathbb{1}_2}^{V=V_0}(Z^*) = -PGP_{\infty}^{V_0}$.*

Proposition 4.1 establishes a first link between the shortage function and PGP approaches. In particular, it demonstrates that the variance fixed shortage function value (see Definition 3.1) computed for the ideal point Z^* with a fixed direction vector with unit coordinates $g = \mathbb{1}_2$ is equal to minus the particular PGP formulation $-PGP_{\infty}^{V_0}$ defined above.

4.2. A MS section suffices to reconstruct the MVS frontier under shorting and a risk-free asset

It is well-known in the MV portfolio model that the combined assumptions of the availability of a risk-free asset and shorting lead to a linear relation between return and normalized variance, because return and normalized risk can be rescaled at will (see, e.g., Balbás et al., 2010, Remark 7). Based on casual inspection of Fig. 1b, one could conjecture that also a linear relationship prevails in the normalized MVS world. Indeed, for the given unit normalized variance, any combination of return and normalized skewness along the normalized MS section spans a line with the risk-free point containing frontier points. The rationale behind this phenomenon is similar to the one mentioned above. While some intuitions underlying this result must be around in the literature (e.g., Hafner and Wallmeier, 2008 on page 161), we are unaware of any precise statement in the multi-moment portfolio literature similar to ours. We formalize this intuition in the following proposition.

Proposition 4.2. *Assume the presence of a risk-free asset and the possibility of short selling this same risk-free asset. Then, the following statements hold true:*

(a) An arbitrary risky portfolio (i.e., with non-zero variance) can be transformed to a unit variance portfolio such that excess return, normalized variance and normalized skewness are proportional for both portfolios;
(b) Conversely, a unit variance portfolio can be transformed to a portfolio with arbitrary strictly positive variance such that excess return, normalized variance and normalized skewness are proportional for both portfolios;
(c) The normalized MVS frontier takes the shape of a cone with vertex the risk-free asset;
(d) To generate the normalized MVS frontier, it suffices to generate a planar section of this frontier not going through the risk-free point and to construct the cone over this intersecting curve with vertex the risk-free asset.

We add three comments to this proposition which generalizes the well-known one fund separation result. First, the presence of a risk-free asset in Proposition 4.2 is essential. Without a risk-free asset, the vertex point of the cone is not identified. Second, note that short selling of all assets is not necessary (as some authors in the PGP literature claim: see, e.g., Chunhachinda et al. (1997) on page 147). Third, the short selling assumption on the risk-free asset is only required to guarantee the existence of the new portfolio $(\bar{w}, \bar{w}_{Rf})$ mentioned in the proof of Proposition 4.2 in Appendix A since $\bar{w}_{Rf}$ might be negative depending on the given data. However, if $\bar{w}_{Rf}$ is positive in a particular case where short selling is excluded, then part (a) of Proposition 4.2 still holds true. In terms of visualization, we can then observe that the normalized MVS frontier contains a partial cone. In Figure B.11b in Appendix B, this phenomenon is clearly noticeable near the risk-free asset.

The key advantage of Proposition 4.2 (part (d)) is that it provides a new method for geometrically reconstructing the normalized MVS frontier from a two-dimensional normalized MS section obtained at unit variance level. Moreover, because of the straightforward relations between normalized and non-normalized coordinates, a normalized MVS frontier can be easily transformed into a non-normalized MVS frontier (and reverse). This new reconstruction technique is illustrated in Appendix B in Figure B.1 visualizing the same frontier as the one in Fig. 1, except that it is reconstructed from the two-dimensional MS section in Figure B.2. Apart from the generalization of the traditional one fund separation results (see, e.g., Luenberger, 1998, Chapter 6) to the MVS portfolio model, it is clear that the possibility of generating a complete MVS frontier from a simple MS section saves computer time.[14]

5. Conclusions

In this contribution, a first attempt is made to bridge the gap between two seemingly different approaches for determining MVS optimal portfolios. We are now in a position to summarize the main contributions.

First, we clarified some results known in the GP literature in OR but seemingly leading to ambiguities in the PGP applications of the Lai (1991) model in finance (similar to Gan (2001) who points out other pitfalls in this PGP literature). This leads to a focus on a geometric interpretation of the Lai (1991) contribution. First, PGP points are located on an unit variance MS section that is part of a MVS portfolio set reconstructed via a shortage function (following Kerstens et al., 2011). This MS section can also be reconstructed using a new, variance fixed shortage function. Second, these PGP points on the unit variance MS section of the MVS frontier are MVS efficient. Third, the PGP approach reconstructs the MS section starting from an 'ideal' portfolio with maximal return and maximal skewness situated outside the portfolio frontier. In particular, the MVS image of a PGP optimal portfolio is locally a tangency point of a PGP circle and the unit variance MS section of the MVS frontier. Subject to some remarks, the PGP approach is normally capable of reconstructing the same MS section of the portfolio frontier. Fourth and finally, the claim in Lai (1991) that PGP also works with a risk-free asset and no shorting is incorrect. However, generalizations of this PGP approach allowing for more flexible portfolio weights are possible by restricting the variance constraint within the empirical range.

Second, we develop two new theoretical results. First, the variance fixed shortage function with a fixed direction vector with unit coordinates is equal to a limiting case of the PGP model of Lai (1991). This first result relating hitherto different portfolio literatures is but a first step to investigating any eventual additional relations among the multitude of different MVS portfolio approaches. Second, we demonstrate that a single MS section (inferred from PGP points, or from the variance fixed shortage function) is sufficient to reconstruct the MVS frontier in the presence of a risk-free asset and its shorting. This generalized one fund separation result offers a reconstruction strategy that definitely saves computer time. Furthermore, it also offers a strong ex-post justification for the basic Lai (1991) model in this particular portfolio setting.

We see two direct challenges following this investigation. First, it would be good if some PGP models could deliver a procedure for three-dimensional MVS frontier reconstruction in general portfolio settings (which is readily available for the shortage function). Apart from a portfolio setting with shorting and a risk-free asset, two-dimensional sections from an otherwise unknown three-dimensional MVS frontier model are of limited value for portfolio management. Three-dimensional MVS frontiers deliver much more information. This calls for a generalization of the current PGP approach capable of making three-dimensional MVS frontier reconstructions (where perhaps the Leung et al. (2001) or Gan (2001) frameworks offer some perspectives). Second, while the duality between shortage function and MVS utility function has been firmly established (Briec et al., 2007), the link between the powers α and β in the current PGP formulations and investor preferences remains

[14] Just to provide some idea: while it takes more than 5000 seconds to reconstruct Fig. 1 with Maple version 14, one needs less than 100 seconds for Figure B.1 available in Appendix B. Both figures are created on a Dell Latitude D610 with 4 Gb RAM.

somewhat underdeveloped. Indeed, Gan (2001) explicitly illustrates that the current PGP formulations imply quite rudimentary equivalent multiple-objective utility functions. This calls for further refinements linking the Lai (1991) model and its variations to developments in the MCDA and GP literatures (see, e.g., Miettinen, 1999 or Spronk et al., 2005, among others).

Wrapping up, this first attempt to bridge the gap between two seemingly unrelated approaches to MVS portfolio modeling finds quite some common ground. Obviously, there is still a large variety of alternative approaches around for which family resemblances remain to be identified. However, this is a fruitful avenue for future research. From a more practical point of view, we think the main contribution lies in the new way of reconstructing MVS portfolio sets from two-dimensional sections. This new way leads to substantial gains in computer time in a portfolio setting with a risk-free asset and shorting. But, on top of this it is obvious to extend both the new, variance fixed shortage function and the new, generalized PGP model to define any two-dimensional section of the MVS portfolio frontier.

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Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.ejor.2013.04.021.

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