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Input Efficiency Measures: A Generalized, Encompassing Formulation

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Abstract. This contribution defines a new generalized input efficiency measure which encompasses and thus links four well-known input efficiency measures: the Debreu-Farrell measure, the Färe-Lovell measure, the asymmetric Färe measure, and the multiplicative Färe-Lovell measure. The axiomatic properties of this new measure are studied. The generalized input efficiency measure naturally leads to the definition of new measures as special cases. It also provides a general framework for testing the choice of efficiency measures. Examples of mathematical programming models in specific cases are established to illustrate this new measure.

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1. Introduction

Debreu (1951) and Farrell (1957) described the first concept of a radial input efficiency measure. Their seminal work has since then been extended to several other nonradial efficiency measures.

In particular, Färe and Lovell (1978) defined a new Färe-Lovell input efficiency measure as an arithmetic mean of its component measures (sometimes also known as a Russell efficiency measure). This measure was defined to reconcile efficiency measures with the notion of Koopmans (1951) efficiency (see also Russell 1985, 1988). A generalization of the Färe-Lovell efficiency measure allowing for a different weighting of each dimension has been proposed in Thanassoulis and Dyson (1992), Zhu (1996), and Ruggiero and Bretschneider (1998).

Another asymmetric Färe input efficiency measure looks for the minimum over its component measures (see, e.g., Färe 1975, Färe et al. 1983). It is the basis for further developments looking at, for instance, the problem of finding the shortest path to the efficient subset (e.g., González and Álvarez 2001). Note that Kopp (1981a, b) is very critical about both the asymmetric Färe and the Färe-Lovell efficiency measures. An early survey of this literature is found in Färe et al. (1983).

Finally, we include a multiplicative version of the Färe-Lovell efficiency measure which belongs to this same family of efficiency measures. It can also be interpreted as an input-oriented special case of the nonoriented geometric distance function proposed in Portela and Thanassoulis (2005, 2007).

Most of this literature and especially the axiomatic properties satisfied by these various input efficiency measures on general technologies has been surveyed in Russell and Schworm (2009). It should be noted that in the operational research literature some axioms related to computational issues like unit and translation invariance have been discussed (see, e.g., Lovell and Pastor 1995), which are almost absent in the economic literature.

The basic economic motivation of this literature on efficiency measures is linked to the observation that the traditional radial efficiency measure projects on the isoquant and that a substantial number of slacks and surpluses may appear. This fact reveals that a substantial number of observations are situated close to the weak efficient subset and the isoquant rather than close to the efficient subset of the technology. However, recently Briec et al. (2018) reported a remarkably high incidence of congestion (an extreme form of technical inefficiency) in several studies published in the literature. This fact seems to indicate that in some samples quite a few observations are situated close to the isoquant and boundary of the input set of the technology. If the latter study is corroborated and it turns out that substantial amounts of observations are situated away from the efficient subset, then this may call for another focus on efficiency measurement. Indeed, ideally one would like to have a framework within which efficiency is measured relative to the subset of the technology where the majority of observations are situated. It is the main purpose of this article to provide such a framework.

In this article, we define a new, generalization of these input efficiency measures that clearly demonstrates the links between the four previously cited measures.[1] Furthermore, we explore the axiomatic properties of this new input efficiency measure, as well as the ways it can be computed using nonparametric specifications of technology. Our new measure therefore offers an encompassing framework for the main existing efficiency measures in the economic literature that turn out to be just special limiting cases of the new one.

Furthermore, in line with the economic literature, this article mainly concentrates on input efficiency. Obviously, these results can be transposed to other measurement orientations (i.e., output and graph). In fact, some of the above-mentioned efficiency measures have also been defined for other measurement orientations (see, for instance, Thanassoulis and Dyson 1992, Portela and Thanassoulis 2007).

2. Assumptions on Technology and Definitions of Efficiency Measures

Technology describes all production possibilities to transform input vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ into output vectors $y = (y_1, \ldots, y_m) \in \mathbb{R}^m_+$: $T = \{(x,y) : x \text{ can produce } y\}$. Given our focus on input-oriented efficiency measurement, this technology can be represented by its input sets:

$$L(y) = \{x \in \mathbb{R}^n_+ : x \text{ can produce } y\}. \tag{1}$$

Occasionally, we also need the output set associated with technology T. It denotes all output vectors $y \in \mathbb{R}^m_+$ that can be produced from a given input vector $x \in \mathbb{R}^n_+$: $P(x) = \{y : (x,y) \in T\}$.

The following standard conditions are imposed on $L(y)$ (see, e.g., Hackman 2008 for details):

Assumption 1. $L(0) = \mathbb{R}^n_+$ and $y \neq 0 \Rightarrow 0 \notin L(y)$;

Assumption 2. let $y_k \in \mathbb{R}^n_+, k \geq 0$ such that $\lim_{k\to\infty} y_k = +\infty$; then $\bigcap_{k\to\infty} L(y_k) = \emptyset$;

Assumption 3. $L(y)$ is a closed set;

Assumption 4. $x \in L(y)$ and $x' \geq x \Rightarrow x' \in L(y)$;

Assumption 5. $y \in \mathbb{R}^m_+$ and $v \geq y \Rightarrow L(v) \subset L(y)$.

Apart from the traditional regularity assumptions (possibility of inaction, boundedness, and closedness), Assumptions 4 and 5 represent the strong or free disposability of inputs and outputs, respectively. We remark that we do not impose any convexity assumption on the input sets. These axioms are almost as weak as the ones proposed in the early literature on efficiency measures (see, e.g., Färe and Lovell 1978, Russell 1985), except that these authors weakened the disposability axioms.

When discussing input efficiency measures, it is important to distinguish three subsets of the input set. First, we can define the isoquant of an input set as

$$\operatorname{Isoq} L(y) = \{x \in L(y) : \lambda x \notin L(y), \forall \lambda \in [0,1[\}. \tag{2}$$

Next, the weak efficient subset is defined by

$$\operatorname{WEff} L(y) = \{x \in L(y) : u < x \Rightarrow u \notin L(y)\}. \tag{3}$$

Finally, the strong efficient subset of an input set is defined as

$$\operatorname{Eff} L(y) = \{x \in L(y) : u \leq x \text{ and } u \neq x \Rightarrow u \notin L(y)\}. \tag{4}$$

Obviously, these three subsets of the input set are related to one another: $\operatorname{Eff} L(y) \subseteq \operatorname{WEff} L(y) \subseteq \operatorname{Isoq} L(y) \subseteq L(y)$.

Now we can recall the definition of the Debreu (1951) and Farrell (1957) radial efficiency measure $DF : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+ \cup \{-\infty, \infty\}$ as follows:

$$DF(x,y) = \begin{cases} \inf_{\delta \in \mathbb{R}_+} \{\delta : \delta x \in L(y)\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

This radial efficiency measure indicates the maximal equiproportionate reduction in all inputs which still allows production of the given output vector on the isoquant of the input set.

From a debate on axiomatic properties of radial efficiency measures, the Färe and Lovell (1978) efficiency measure emerged. This function $FL : \mathbb{R}^n_+ \backslash \{0\} \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ can be defined as follows:

$$FL(x,y) = \begin{cases} \inf_{\beta \in \mathbb{R}^n_+} \left\{ \frac{1}{|I(x)|} \sum_{i \in I(x)} \beta_i : \beta \odot x \in L(y), \beta_i \in [0,1] \right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise,} \end{cases} \tag{6}$$

where $\odot$ denotes the Hadamard product (element by element) of two vectors, and for all $x \in \mathbb{R}^n_+$ the support of x is defined as $I(x) = \{i : x_i > 0\}$. This Färe and Lovell (1978) efficiency measure indicates the minimum average sum of dimension-wise reductions in each input dimension which maintain production of given outputs on the efficient subset of the input set. In line with, for example, Ruggiero and Bretschneider (1998) and Zhu (1996), one can also define a weighted Färe-Lovell efficiency measure as follows. For all $\alpha \in \mathbb{R}^n_{++}$, one defines the function: $FL_\alpha : \mathbb{R}^n_+ \backslash \{0\} \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ as

$$FL_\alpha(x,y) = \begin{cases} \inf\limits_{\beta \in \mathbb{R}^n_+} \left\{ \frac{1}{|I(x)|} \sum\limits_{i \in I(x)} (\alpha_i \beta_i) : \beta \odot x \in L(y), \beta_i \in [0,1] \right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise,} \end{cases} \tag{7}$$

where β is the vector of $\mathbb{R}^{I(x)}$ whose elements are β_i for $i \in I(x)$, and $\alpha \in \mathbb{R}^{I(x)}_{++}$ is the vector whose elements are α_i for $i \in I(x)$ and such that $\sum_{i \in I(x)} \alpha_i = 1$.

We also recall the asymmetric Färe (1975) input efficiency measure defined by $AF : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+ \cup \{\infty\}$,

$$AF(x,y) = \begin{cases} \min\limits_{i \in I(x)} \inf\{ \beta_i : (\mathbb{1} + (\beta_i - 1)e_i) \odot x \in L(y)\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise.} \end{cases} \tag{8}$$

This asymmetric Färe input efficiency measure takes the minimum of the dimension-wise reductions in each input dimension which allow production of given outputs on the boundary of the input set.

Paralleling Färe and Lovell (1978) and in line with Portela and Thanassoulis (2005), Ruggiero and Bretschneider (1998), and Zhu (1996), one can define a weighted multiplicative (rather than additive) Färe-Lovell type of input efficiency measure. For all $\alpha \in \mathbb{R}^n_{++}$, let us define this function as follows: $\mathrm{MFL}_\alpha : \mathbb{R}^n_+ \backslash \{0\} \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ as

$$\mathrm{MFL}_\alpha(x,y) = \begin{cases} \inf\limits_{\beta \in \mathbb{R}^n_+} \left\{ \prod\limits_{i \in I(x)} (\beta_i)^{\alpha_i} : \beta \odot x \in L(y), \beta_i \in [0,1] \right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise,} \end{cases} \tag{9}$$

where β and α are restricted like the weighted Färe-Lovell efficiency measure (7). Since this function is the weighted multiplicative equivalent of the Färe and Lovell (1978) measure of input efficiency, it is termed a weighted multiplicative Färe-Lovell input efficiency measure. It also projects an observed input-output combination on the efficient subset of the input set. This weighted multiplicative Färe-Lovell measure includes as a special case the multiplicative Färe-Lovell measure proposed by Portela and Thanassoulis (2005) setting $\alpha_i = \frac{1}{|I(x)|}$ for all $i \in I(x)$. In the remainder, this multiplicative (unweighted) Färe-Lovell measure will be denoted MFL: $\mathrm{MFL}(x,y) = \mathrm{MFL}_\alpha(x,y)$ in the case where $\alpha_i = 1$ for all i.

As shown in the next section, these four input efficiency measures and their two variations turn out to belong to a single family.

3. A New Generalized Input Efficiency Measure and Its Limiting Special Cases

3.1. Generalized Sum and Generalized Measures

For all $p \in]0, +\infty[$, let $\phi_p : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be the map defined by $\phi_p(\lambda) = \lambda^p$. For all $p \neq 0$, the reciprocal map is $\phi_p^{-1} := \phi_{\frac{1}{p}}$. First, it is quite straightforward to state that (i) ϕ_p is defined over $\mathbb{R}_+$; (ii) ϕ_p is continuous over $\mathbb{R}_+$; and (iii) ϕ_p is bijective over $\mathbb{R}_+$. Second, let us focus on the case $p \in]-\infty, 0[$. The map $x \mapsto x^p$ is not defined at point $x = 0$. Thus, it is not possible to construct a bijective endomorphism on $\mathbb{R}_+$.

For all $p \in]-\infty, 0[$ we consider the function ϕ_p defined by

$$\phi_p(\lambda) = \begin{cases} \lambda^p & \text{if } \lambda > 0 \\ +\infty & \text{if } \lambda = 0. \end{cases} \tag{10}$$

The reciprocal is the map $\phi_{\frac{1}{p}}$ defined on $\mathbb{R}_{++} \cup \{\infty\}$ as

$$\phi_{\frac{1}{p}}(\lambda) = \begin{cases} \lambda^{\frac{1}{p}} & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = \infty. \end{cases} \tag{11}$$

Let us investigate the ϕ_p-generalized sum analyzed by Ben-Tal (1977). For all $(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+$ and for all $p > 0$ the ϕ_p-generalized sum is given by

$$\sum_{i \in [n]}^{\phi_p} \beta_i := \phi_p^{-1}\left(\sum_{i \in [n]} \phi_p(\beta_i)\right) = \left(\sum_{i \in [n]} (\beta_i)^p\right)^{\frac{1}{p}}. \tag{12}$$

If $p < 0$, using the symbolism $\frac{1}{0} = +\infty$ we have, by construction,

$$\sum_{i \in [n]}^{\phi_p} \beta_i := \phi_p^{-1}\left(\sum_{i \in [n]} \phi_p(\beta_i)\right) = \begin{cases} \left(\sum\limits_{i \in [n]} (\beta_i)^p\right)^{\frac{1}{p}} & \text{if } \min_i \beta_i > 0 \\ 0 & \text{if } \min_i \beta_i = 0. \end{cases} \tag{13}$$

In the next statement, it is shown that this generalized sum is continuous for all $p \in \mathbb{R}\backslash\{0\}$. This is important to provide a general formulation encompassing as a special case both the situations where $p > 0$ and $p < 0$.

Lemma 1. *For all $p \in \mathbb{R}\backslash\{0\}$, the map $\beta \mapsto \sum^{\phi_p}_{i \in [n]} \beta_i$ is continuous over $\mathbb{R}^n_+$.*

The proofs of all lemmas and propositions are in the e-companion.

The Debreu (1951) and Farrell (1957), Färe and Lovell (1978), asymmetric Färe, and multiplicative Färe-Lovell measures are all input efficiency measures that can be interpreted as special cases of a new type of extended Färe-Lovell input efficiency measure. This new generalized Färe-Lovell input efficiency measure can be defined as follows.

Definition 1. *For all* $p \in \mathbb{R}\backslash\{0\}$, *the generalized Färe-Lovell input efficiency measure* $\mathrm{GFL}_p : \mathbb{R}^n_+\backslash\{0\} \times \mathbb{R}^m_+ \longrightarrow [0,1] \cup \{\infty\}$ *is defined by*

$$\mathrm{GFL}_p(x,y) = \begin{cases} \inf\left\{\frac{1}{|I(x)|^{\frac{1}{p}}} \sum_{i\in I(x)}^{\phi_p} \beta_i : \beta \odot x \in L(y); \; \beta_i \in [0,1]\right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise,} \end{cases} \tag{14}$$

where $|I(x)|$ *stands for the cardinality of* $I(x)$.

Clearly, if $p > 0$, then we have

$$\mathrm{GFL}_p(x,y) = \begin{cases} \inf\left\{\left(\sum_{i\in I(x)} \frac{1}{|I(x)|}\beta_i{}^p\right)^{\frac{1}{p}} : \beta\odot x \in L(y); \; \beta_i \in [0,1]\right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise.} \end{cases} \tag{15}$$

Notice that, in Definition 1, we consider the cases where p takes either positive or negative values. In the remainder, the case $p = 2$ is termed the quadratic generalized Färe-Lovell measure, by analogy with the definition of a canonical quadratic form. To parallel this terminology, the case $p = -1$ is termed the harmonic generalized Färe-Lovell measure, by analogy with the definition of the harmonic mean. Clearly, if $p > 1$, then the new measure can be viewed as the ℓ_p Hölder distance from the origin to the subset defined as $\{\beta \in \mathbb{R}^n_+ : \beta \odot x \in L(y)\}$, where $x \in L(y)$. Obviously, a weighted generalized Färe-Lovell efficiency measure can be similarly defined as

$$\mathrm{GFL}_{p,\alpha}(x,y) = \begin{cases} \inf\left\{\frac{1}{|I(x)|^{\frac{1}{p}}} \sum_{i\in I(x)}^{\phi_p} \alpha_i\beta_i : \beta \odot x \in L(y); \; \beta_i \in [0,1]\right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise,} \end{cases} \tag{16}$$

where $\alpha_i > 0$ for all i. However, it must already be mentioned that its link to the multiplicative Färe-Lovell measure is not as clear as it is in the nonweighted case.

In the following it is shown that the generalized Färe-Lovell measure involves an optimal value of β, which allows one to consider a reference input vector. These elementary properties are important for the results established hereafter in the article.

Lemma 2. *Suppose that* $(x,y) \in T$ *and* $x \neq 0$. *For all* $p \in \mathbb{R}\backslash\{0\}$, *we have the following properties:*

a. *The set* $B = \{\beta \in [0,1]^n : \beta \odot x \in L(y)\}$ *is a compact subset of* $\mathbb{R}^n_+$.

b. $\mathrm{GFL}_p(x,y) = \inf\{\frac{1}{|I(x)|^{\frac{1}{p}}} \sum_{i\in I(x)}^{\phi_p}\beta_i : \beta \in B\}$.

c. *There exists* $\beta^\star \in [0,1]^n$ *such that*

$$\mathrm{GFL}_p(x,y) = \frac{1}{|I(x)|^{\frac{1}{p}}} \sum_{i\in I(x)}^{\phi_p} \beta_i^\star$$

and such that $\beta^\star \odot x \in \mathrm{Eff}\, L(y)$.

It seems straightforward to define some similar generalized nonoriented measures. However, as is the case for the output-oriented measure, some problems arise when characterizing the efficient subset. This fact was pointed out in Briec (2000). This is why we remain focused on the input-oriented measures in this contribution.

However, for the sake of argument, let us still provide the definition of a graph generalized measure: given two real numbers $p, q \in \mathbb{R}\backslash\{0\}$, a graph generalized measure $\mathrm{GFL}_{G,p,q} : \mathbb{R}^n_+\backslash\{0\} \times \mathbb{R}^m_+\backslash\{0\} \longrightarrow [0,1] \cup \{\infty\}$ can be defined as

$$\mathrm{GFL}_{G,p,q}(x,y) = \begin{cases} \inf_{\substack{\beta\in[0,1] \\ \gamma\in[1,+\infty[}} \left\{\frac{|J(y)|^{\frac{1}{p}}}{|I(x)|^{\frac{1}{p}}} \left(\sum_{i\in I(x)}^{\phi_p} \beta_i\right)\left(\sum_{j\in J(y)}^{\phi_q} \gamma_i\right)^{-1} : (\beta \odot x, \gamma \odot y) \in T\right\} & \text{if } (x,y) \in T \\ +\infty & \text{otherwise,} \end{cases} \tag{17}$$

where $|J(y)|$ stands for the cardinality of $J(y) = \{j : y_j > 0\}$. This ratio formulation is inspired by the enhanced Russell graph efficiency measure of Pastor et al. (1999) rather than the original additive formulation of the original Russell graph efficiency measure of Färe et al. (1985). By remembering that, given a real parameter $r > 0$, the Chavas and Cox (1999) measure is a map $E_C : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \longrightarrow [0,1] \cup \{\infty\}$ defined by $E_C(x,y) = \inf_\lambda\{(\lambda^{1-r}x, \lambda^r y) \in T\}$ and choosing $p = 1 - r$ and $q = -r$, it follows easily that the Chavas and Cox (1999) measure is a special limiting case of this graph generalized efficiency measure. Other nonoriented versions of these efficiency measures are left for future work.

3.2. Axiomatic Properties

Since this input-oriented generalized Färe-Lovell measure is new, it is important to characterize its axiomatic properties in the following proposition.

Proposition 1. *Under Assumptions 1–4, for all* $p \in \mathbb{R}\backslash\{0\}$, *we have the following:*

a. *If* $x \neq 0$, *then* $\mathrm{GFL}_p(x,y) = 1$ *if and only if* $x \in \mathrm{Eff}\, L(y)$ *(characterization or indication of the efficient subset).*

b. *For all* $y \neq 0$ and $x, u \in L(y)$, $u \geq x$, *and* $x \neq 0$ *imply that* $\mathrm{GFL}_p(u,y) \geq \mathrm{GFL}_p(x,y)$ *(weak monotonicity in the inputs).*

c. *For all* $x \neq 0$ *and* $y, v \in P(x)$, $v \geq y$ *implies that* $\mathrm{GFL}_p(x,v) \geq \mathrm{GFL}_p(x,y)$ *(weak monotonicity in the outputs).*

d. *If* $(x,y) \in T$ *and* $x \neq 0$, *then for all* $\lambda \geq 1$, $\mathrm{GFL}_p(\lambda x, y) \leq \lambda\, \mathrm{GFL}_p(x,y)$ *(subhomogeneity).*

In brief, this generalized Färe-Lovell measure has the same properties as the original Färe-Lovell efficiency measure.

The properties of all other efficiency measures, except the multiplicative Färe-Lovell efficiency measure, are well-known in the literature and do not need any further comments (see the survey in Russell and Schworm 2009). In addition, the multiplicative Färe-Lovell efficiency measure satisfies the following properties.

Proposition 2. *Under Assumptions 1–4, we have the following:*

a. *If* $x \neq 0$, *then* $\mathrm{MFL}_\alpha(x,y) = 1$ *if and only if* $x \in \mathrm{Eff}\, L(y)$ *(characterization or indication of the efficient subset).*

b. *For all* $y \neq 0$ *and* $x, u \in L(y)$, $u \geq x$ *and* $x \neq 0$ *imply that* $\mathrm{MFL}_\alpha(u,y) \geq \mathrm{MFL}_\alpha(x,y)$ *(weak monotonicity in the inputs).*

c. *For all* $x \neq 0$ *and* $y, v \in P(x)$, $v \geq y$ *implies that* $\mathrm{MFL}_\alpha(x,v) \geq \mathrm{MFL}_\alpha(x,y)$ *(weak monotonicity in the outputs).*

d. *If* $(x,y) \in T$ *and* $x \neq 0$, *then for all* $\lambda \geq 1$, $\mathrm{MFL}_\alpha(\lambda x, y) = \lambda^{\sum_{i \in I(x)} \alpha_i} \mathrm{MFL}_\alpha(x,y)$ *(homogeneity).*

We remark that Portela and Thanassoulis (2007) mention some of these properties, but for a nonoriented efficiency measure. Thus, the multiplicative Färe-Lovell efficiency measure has the same properties as the generalized Färe-Lovell and the original Färe-Lovell measures, apart from the fact that it satisfies homogeneity rather than subhomogeneity.

The property of continuity has been introduced in the literature by Russell (1990). In recent work Russell and Schworm (2018) distinguish between two large families of efficiency measures: slack-based and path-based. These authors show that all slack-based efficiency measures satisfy the indication property but violate continuity, whereas all path-based efficiency measures satisfy continuity but violate the indication property. Since the generalized Färe-Lovell input measure clearly belongs to the class of slack-based efficiency measures, given a suitable homeomorphic transformation, it fails continuity.

3.3. Farrell Measure as a Special Limiting Case of the Generalized Färe-Lovell Measure

In fact, especially in the case where $p \geq 1$, the map $\beta \mapsto (\frac{1}{n}\sum_{i=1}^{n}(\beta_i)^p)^{1/p}$ benefits from the properties of an ℓ_p-norm. This formulation in Definition 1 as well as the optimization developed in Section 4 below presents some analogies to the well-known class of goal programming problems which consist in finding an optimal feasible point close to a series of targets (see, for instance, Jones and Tamiz 2010 for a recent overview).

Proposition 3. *Assume that* B *is a compact subset of* $\mathbb{R}^n_+$, *and assume that* $p_0 \in [0, +\infty]$. *We have*

$$\lim_{p \longrightarrow p_0} \inf_{\beta \in B} \left(\frac{1}{n}\sum_{i=1}^{n}(\beta_i)^p\right)^{1/p} = \inf_{\beta \in B} \lim_{p \longrightarrow p_0} \left(\frac{1}{n}\sum_{i=1}^{n}(\beta_i)^p\right)^{1/p}. \quad (18)$$

In particular, we have the following:

a. $$\lim_{p \longrightarrow 0^+} \inf_{\beta \in B} \left(\frac{1}{n}\sum_{i=1}^{n}(\beta_i)^p\right)^{1/p} = \inf_{\beta \in B} \prod_{i=1}^{n}(\beta_i)^{1/n}, \quad (19)$$

b. $$\lim_{p \longrightarrow +\infty} \inf_{\beta \in B} \left(\frac{1}{n}\sum_{i=1}^{n}(\beta_i)^p\right)^{1/p} = \inf_{\beta \in B} \max_{i=1,\ldots,n} \beta_i. \quad (20)$$

The next statement relates the Farrell measure to the map $\beta \mapsto \max_{i \in I(x)} \beta_i$. It is a useful result in the remainder.

Proposition 4. *Under Conditions* 1–4, *for all* $x \in L(y)$, *we have*

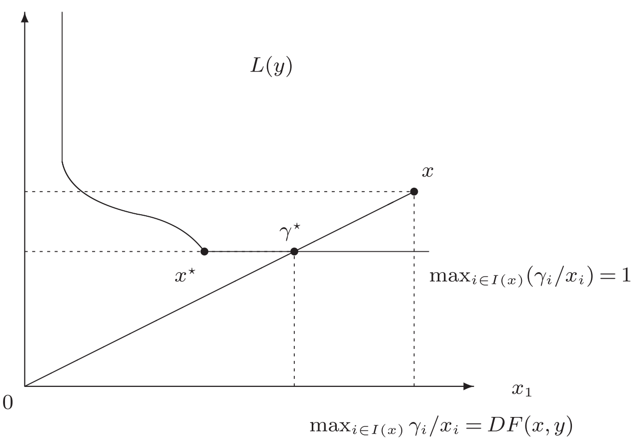
$$DF(x,y) = \inf_{\beta}\left\{\max_{i \in I(x)} \beta_i : \beta \odot x \in L(y), \beta_i \in [0,1]\right\}. \quad (21)$$

To understand the key idea, note that, setting $\gamma = \beta \odot x$, we have

$$\inf_{\beta}\left\{\max_{i \in I(x)} \beta_i : \beta \odot x \in L(y), \beta_i \in [0,1]\right\}$$
$$= \inf_{\gamma}\left\{\max_{i \in I(x)} \gamma_i / x_i : \gamma \in L(y), \gamma \leq x\right\}. \quad (22)$$

Figure 1 illustrates how one can retrieve the Farrell measure from this maximum function. For all real numbers $\delta > 0$, the subset $A_x(\delta) = \{\gamma : \max_i \frac{\gamma_i}{x_i} = \delta\}$ has a supremum element δx. Hence, if $\delta \geq DF(x,y)$, then $A_x(\delta) \cap L(y) \neq \emptyset$. Set $\gamma^\star = DF(x,y)$. This figure depicts the fact that there exists some $\beta^\star$ with $x^\star = \beta^\star \odot x$ (which

Figure 1. New Formulation of the Debreu-Farrell Measure



implies that $\beta_i^\star = \frac{x_i^\star}{x_i}$ for all $i \in I(x)$) and $\max_{i\in I(x)} \beta_i^\star = \max_{i\in I(x)} x_i^\star/x_i = \max_{i\in I(x)} \gamma_i^\star/x_i = DF(x,y)$.

We now come to one of the central results of this contribution showing that both the Farrell and the multiplicative Färe-Lovell measures can be viewed as limiting cases of the generalized Färe-Lovell measure.

Proposition 5. *Under Assumptions 1–4, for all $(x,y) \in T$ and $x \neq 0$ we have the following:*

a. $\mathrm{GFL}_1(x,y) = FL(x,y)$.
b. $\lim_{p\longrightarrow 0^+} \mathrm{GFL}_p(x,y) = \mathrm{MFL}\,(x,y)$.
c. $\lim_{p\longrightarrow +\infty} \mathrm{GFL}_p(x,y) = DF(x,y)$.

Notice that there is no evidence that the limit of the weighted generalized Färe-Lovell measure is the weighted multiplicative measure when p tends toward 0.

Proposition 1 established that the generalized Färe-Lovell measure satisfies the axiomatic properties inherited from the standard case $p = 1$. However, it is well known that the Debreu-Farrell measure (which corresponds to the case $p = \infty$) does not allow one to characterize the strong efficient subset. This is due to the fact that the map $\beta \mapsto \max_{i\in I(x)} \beta_i$ involves an idempotent algebraic structure. (This is not the case of the generalized sum.) Therefore, its minimization does not impose a transfer between the inputs required to reach the strong efficient subset.

Lemma 3. *Under Assumptions 1–4, for all $(x,y) \in T$, if $x \neq 0$, we have the following properties:*

a. *There is some $\beta^0 \in [0,1]^n$ such that*

$$\mathrm{MFL}\,(x,y) = \prod_{i\in I(x)} (\beta_i^0)^{\frac{1}{|I(x)|}} \tag{23}$$

and such that $\beta^0 \odot x \in \mathrm{Eff}\, L(y)$.

b. *There is some $\beta^\infty \in [0,1]^n$ such that*

$$DF(x,y) = \max_{i\in I(x)} \beta_i^\infty \tag{24}$$

and such that $\beta^\infty \odot x \in \mathrm{Eff}\, L(y)$.

3.4. Generalized Färe-Lovell Measure and Asymmetric Färe Measure

In this subsection we establish a relation between the asymmetric Färe measure and the generalized Färe-Lovell measure. To do this, we focus on the case $p < 0$.

Lemma 4. *Assume that B is a compact subset of $\mathbb{R}_+^n$. For all $p_0 < 0$, we have*

$$\lim_{p\longrightarrow p_0} \inf_{\beta\in B} \left(\frac{1}{n^{\frac{1}{p}}} \sum_{i\in[n]}^{\phi_p} \beta_i\right) = \inf_{\beta\in B} \lim_{p\longrightarrow p_0} \left(\frac{1}{n^{\frac{1}{p}}} \sum_{i\in[n]}^{\phi_p} \beta_i\right). \tag{25}$$

In particular, we have

$$\lim_{p\longrightarrow -\infty} \inf_{\beta\in B} \left(\frac{1}{n^{\frac{1}{p}}} \sum_{i\in[n]}^{\phi_p} \beta_i\right) = \inf_{\beta\in B} \min_{i=1,\ldots,n} \beta_i. \tag{26}$$

The above statement extends Proposition 3 to the case where $p < 0$. Proposition 6 is then an immediate consequence.

Proposition 6. *Under Assumptions 1–4, for all $(x,y) \in T$ if $x \neq 0$, we have*

$$\lim_{p\longrightarrow -\infty} \mathrm{GFL}_p(x,y) = \inf_{\beta} \left\{ \min_{i\in I(x)} \beta_i : \beta \odot x \in L(y), \beta_i \in [0,1] \right\}. \tag{27}$$

The next statement parallels Proposition 4 in the context of the asymmetric Färe measure.

Proposition 7. *Under Assumptions 1–4, for all $(x,y) \in T$, we have*

$$A(x,y) = \min_{i\in I(x)} \inf_{\beta} \{\beta_i : \beta \odot x \in L(y), \beta_i \in [0,1]\}. \tag{28}$$

Figure 2 illustrates the key intuition of this result. The asymmetric Färe measure is defined as an optimization over the intersection of the reverse free-disposal cone and the input set. Notice that this is not true if the free disposal assumption does not hold.

The next statement completes Proposition 5 in the case where $p < 0$, establishing a relation to the asymmetric-Färe measure.

Proposition 8. *Under Assumptions 1–4, for all $(x,y) \in T$ we have*

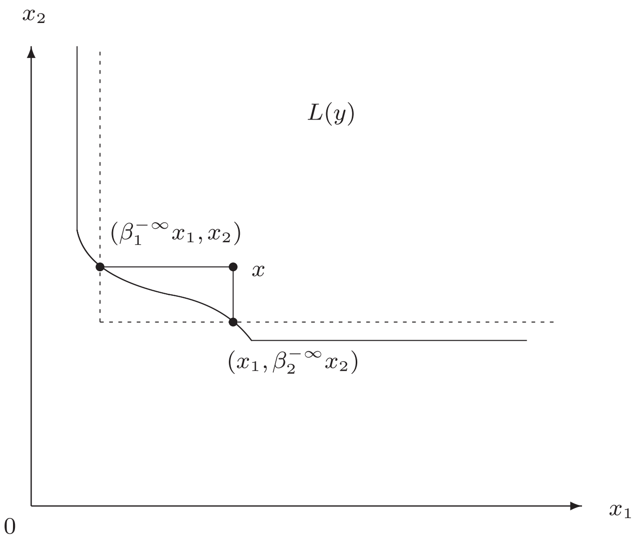
$$\lim_{p\longrightarrow -\infty} \mathrm{GFL}_p(x,y) = AF(x,y). \tag{29}$$

Proposition 9. *Under Assumptions 1–4, for all $(x,y) \in T$ there is some $\beta^{-\infty} \in [0,1]^n$ such that*

$$AF(x,y) = \min_{i\in I(x)} \beta_i^{-\infty} \tag{30}$$

and such that $\beta^{-\infty} \odot x \in \mathrm{Eff}\, L(y)$.

Figure 2. New Formulation of the Asymmetric Färe Measure



Thus, the four efficiency measures discussed in Section 2 are clearly limiting cases of the new generalized Färe-Lovell measure in Definition 1.

Finally, it is rather trivial to a priori order most of these efficiency measures. Indeed, although it is obvious that $DF(x,y) \geq FL(x,y) \geq \text{MFL}(x,y) \geq AF(x,y)$, the new generalized Färe-Lovell input efficiency can also be related to these existing efficiency measures.

Proposition 10. *The generalized Färe-Lovell input efficiency can be ordered as follows for all $(x,y) \in T$ with $x \neq 0$. If $q \geq p > 0$, then*

$$DF(x,y) \geq \text{GFL}_q(x,y) \geq \text{GFL}_p(x,y) \geq FL(x,y) \geq \text{MFL}(x,y). \tag{31}$$

Suppose that $0 > p' > q'$. Then

$$\text{MFL}(x,y) \geq \text{GFL}_{p'}(x,y) \geq \text{GFL}_{q'}(x,y) \geq AF(x,y). \tag{32}$$

In Figure 3, the projection points depend on p. In the case $p = +\infty$, the projection is radial. If $p = -\infty$, then we retrieve the maximal possible contraction of each input that is measured by the asymmetric Färe measure. The case $p = 1$ yields the efficient reference point derived from the Färe-Lovell measure. The intermediate case $p = 0$ corresponds to the multiplicative case.

Clearly, the Debreu-Farrell and asymmetric Färe measures do not yield an efficient reference point. However, one can associate to each one a reference point for p sufficiently large ($p \to \infty$) or small ($p \to -\infty$). Along this line, we say that $\bar{x}^{(p)}$ is a Debreu-Farrell-(ϵ, p) efficient reference point of x if there exists some $\bar{\beta}^{(p)} \in [0,1]^n$ such that $\bar{x}^{(p)} = \bar{\beta}^{(p)} \odot x$ with $(\frac{1}{|I(x)|}\sum_{i=1}^n (\bar{\beta}_i^{(p)})^p)^{1/p} = \text{GFL}_p(x,y)$ and

$$\left|\left(\frac{1}{n}\sum_{i=1}^n \left(\bar{\beta}_i^{(p)}\right)^p\right)^{1/p} - DF_p(x,y)\right| < \epsilon. \tag{33}$$

The basic idea behind this definition is to match each input vector with an efficient reference point that is consistent with the Debreu-Farrell measure for some degree of approximation. Paralleling this definition, we say that $\bar{x}^{(p)}$ is an asymmetric Färe-(ϵ, p) efficient reference point of x if there exists some $\bar{\beta}^{(p)} \in [0,1]^n$ such that $\bar{x}^{(p)} = \bar{\beta}^{(p)} \odot x$ with $\frac{1}{|I(x)|^{\frac{1}{p}}}\sum_{i\in I(x)}^{\phi_p} \beta_i^{(p)} = \text{GFL}_p(x,y)$ and

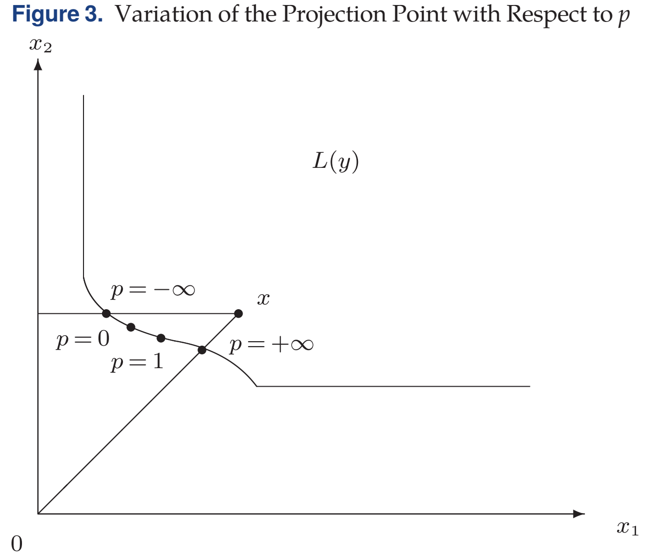
$$\left|\frac{1}{|I(x)|^{\frac{1}{p}}}\sum_{i\in I(x)}^{\phi_p} \beta_i^{(p)} - AF_p(x,y)\right| < \epsilon. \tag{34}$$

Similarly, this definition allows one to match each input vector with an efficient reference point that is consistent with the asymmetric-Färe measure when $p \to -\infty$.

3.5. Prices and Duality

In the following, it is shown that one can provide a dual interpretation of the generalized Färe-Lovell measure in terms of prices. This we do by showing that, in the case where $p \geq 1$, the generalized Färe-Lovell measure can be interpreted as the projection of the origin onto a suitable restriction of the input set. By applying the Nirenberg theorem, a duality result is obtained. Along this line a dual formulation of the Färe-Lovell measure is proposed, and it is shown that

Figure 3. Variation of the Projection Point with Respect to p



the dual properties of the Debreu-Farrell measure appear as a limiting case.

Notice that one can equivalently write

$$\mathrm{GFL}_p(x,y) = \begin{cases} \inf\left\{ \frac{1}{|I(x)|^{\frac{1}{p}}} \sum_{i\in I(x)}^{\phi_p} \frac{u_i}{x_i} : u \in L(y), \right. \\ \left. \quad u \le x \right\} & \text{if } x \in L(y) \\ +\infty & \text{otherwise.} \end{cases} \tag{35}$$

In the case where $p \ge 1$, it is interesting to see that one can provide a dual interpretation based upon the Nirenberg theorem.[2] Let us denote $L_x(y) = \{u \in L(y) : u \le x\}$, which is the set of all the input vectors that are dominated by x and can produce y. Let us define as $C_x : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+$ the function which yields the minimum cost for all the input vectors of $L_x(y)$. Namely, $C_x(w,y) = \inf\{w.u : u \in L_x(y)\}$. For example, if $x \in \mathbb{R}^n_{++}$, then the map $\|\cdot\|_{x^{-1},p} : u \mapsto \frac{1}{n^{\frac{1}{p}}}(\sum_{i\in[n]} |\frac{u_i}{x_i}|^p)^{\frac{1}{p}}$ defines a weighted norm on $\mathbb{R}^n$. If $p = \infty$, then $\|u\|_{x^{-1},\infty} = \max_{i\in[n]} \frac{|u_i|}{x_i} = \lim_{p\longrightarrow\infty} \|u\|_{x^{-1},p}$. It follows that $\mathrm{GFL}_p(x,y) = d_{x^{-1},p}(0, L_x(y)) = \inf\{\|u\|_{x^{-1},p} : u \in L_x(y)\}$, where $d_{x^{-1},p}$ is the distance induced by this norm. The dual norm is then defined by $\|v\|_{x,q} = n^{\frac{1}{p}}(\sum_{i\in[n]} |x_i v_i|^q)^{\frac{1}{q}}$ with $\frac{1}{p} + \frac{1}{q} = 1$, where by definition $\|v\|_{x,q} = \sup\{|v.u| : \|u\|_{x^{-1},p} = 1\}$. The following result is established in the general case where the input vector may have some null components. (The proof is given in the e-companion.)

Theorem 1. *Under Assumptions 1–4, for all $(x,y) \in T$, if $p \ge 1$ and $x \ne 0$, we have the following properties:*

a. $\mathrm{GFL}_p(x,y) = \sup_w\{C_x(w,y) : |I(x)|^{\frac{q-1}{q}}(\sum_{i\in[n]}|x_i w_i|^q)^{\frac{1}{q}} = 1\}$, *with* $\frac{1}{p} + \frac{1}{q} = 1$.

b. $FL(x,y) = \sup_{w\ge 0}\{C_x(w,y) : |I(x)| \max_{i\in[n]}\{x_i w_i\} = 1\}$.

c. $DF(x,y) = \sup_{w\ge 0}\{C_x(w,y) : \sum_{i\in[n]} w_i x_i = 1\}$.

The interpretation of the duality results established in Theorem 1, (a) and (b) is similar to that arising in the Debreu-Farell case. Consider for any input vector a suitable cost function restricted to the dominated input vectors. The generalized Färe-Lovell measure can then be expressed as the maximum ratio between this cost function and the normalized cost of the input vector whose efficiency is measured. The key difference between all these measures is the normalization constraint implied. If $p \in [1, \infty[$, then the problem of computing the generalized Färe-Lovell measure boils down to minimizing a smooth ℓ_p-norm that selects an efficient point on the frontier. In the Färe-Lovell case, the normalization condition of the shadow prices is less restrictive and allows one to modify each input price in coordinate directions to reach an efficient point. This is not the case for the Debreu-Farrell measure for which the additive nature of the normalization constraint imposes a radial projection of any input vector. Notice finally that, in the Debreu-Farrell case, the shadow prices $w^\star$ that are solutions to the dual problem achieve the minimum of the cost function at point $x^\star = DF(x,y)x$. It follows that $C_x(w^\star, y) = C(w^\star, y)$, and we retrieve the standard duality result as a special case.[3]

We now turn to the more practical issue of how this new generalized Färe-Lovell input efficiency measure can be empirically implemented.

4. Generalized Färe-Lovell Efficiency Measure: Mathematical Programming Models

4.1. General Case

We first notice that the map $M_p : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ defined by $M_p(\beta) = (\frac{1}{n}\sum_{i=1}^n (\beta_i)^p)^{1/p}$ is convex for all $p \ge 1$ and concave for all $p \in]0,1[$.

In the following, we limit ourselves to deterministic nonparametric technology specifications. In particular, we consider a variable returns-to-scale technology with strong disposal of inputs and outputs (see, e.g., Banker et al. 1984). Obviously, any other standard nonparametric reference technology could have been selected (see, e.g., Hackman 2008).

Let $\mathcal{J}$ be a finite subset of $\mathbb{N}\backslash\{0\}$, and suppose that we consider a set $A = \{(x_j, y_j) : j \in \mathcal{J}\}$ of $|\mathcal{J}|$ firms. Then, the new generalized Färe-Lovell measure from Definition 1 can be computed for each observation $j \in \mathcal{J}$ as follows:

$$\begin{aligned} \mathrm{GFL}_p(x,y) = \min_{\beta_i, z_i} & \left(\frac{1}{n}\sum_{i=1}^n (\beta_i)^p\right)^{1/p} \\ & \beta_i x_i \ge \sum_{j\in\mathcal{J}} z_j x_{i,j}, \quad i = 1\ldots,n, \\ & y \le \sum_{j\in\mathcal{J}} z_j y_j, \\ & \sum_{j\in\mathcal{J}} z_j = 1, \beta_i \le 1, \beta_i, z_j \ge 0, \end{aligned} \tag{36}$$

where β_i denotes the input efficiency measure component and z_j represents the activity vector.

Depending on the value of p, one can distinguish between several cases with alternative solution strategies for this mathematical programming model. First, in the limiting case $p = -\infty$, $\mathrm{GFL}_{-\infty}(x,y)$ is the asymmetric Färe measure: computing a finite number of N linear programs and taking the minimum over each of these one-dimensional efficiency measures yields the solution. Second, if $p = +\infty$, then $\mathrm{GFL}_\infty(x,y)$ coincides with the standard radial efficiency measure (see Banker et al. 1984) and can be computed by standard linear programming methods. Third, in the case $p \geq 1$, standard convex programming algorithm methods can be used, because the problem boils down to minimizing a convex function over a polyhedral convex set. Notice that we can also compute the efficiency scores by enumeration methods when replacing in the above program the condition $z_j \geq 0$ with $z_j \in \{0, 1\}$: in such a case, we obtain the *FDH* technology.

If $p \in [0, 1[$, then the problem is much more difficult and consists in finding the global minimum of a concave function over a closed, convex set in $\mathbb{R}^n$. The intrinsic difficulty of this problem is due to the fact that a local minimum of the objective function may fail to be a global one. This makes conventional methods of local optimization almost useless (see, e.g., Hoffman 1981, Tuy et al. 1985).

Finally, if $p < 0$, then the difficulties are equivalent because the problem consists in finding the global maximum of a convex function over a closed, convex set in $\mathbb{R}^n$.

In empirical applications, one ideally would like to have some guidelines to determine the optimal choice of the optimal efficiency measure (i.e., the choice of parameter p in the generalized Färe-Lovell efficiency measure). We explore some possibilities.

A first option could consist in a priori choosing one of the subsets of the input set depending on the strength of the efficiency notion one wishes to impose. For instance, if the strong efficient subset is privileged, then the choice is limited between the Färe-Lovell, the multiplicative Färe-Lovell, and the generalized Färe-Lovell efficiency measures.

Another possibility is to look for convenience and insist on a solution by means of linear programs (LPs), a concern that also implicitly or explicitly permeates part of the nonparametric literature. In this case, in addition to the possibilities listed above, we can offer two more specific results where the generalized Färe-Lovell efficiency measure and the multiplicative Färe-Lovell efficiency measure can be solved via LPs for some particular specifications of technology. We discuss each of these special cases below in Sections 4.2 and 4.3, respectively.

A final option is to look at the choice of technology specification and transpose strategies from this context to the choice of the efficiency measure. It is well known that technical efficiency measures are very sensitive to the choice of functional specification in parametric frontier methods: a more flexible specification detects lower amounts of inefficiency (see Kopp and Smith 1980, Giannakas et al. 2003). Also for nonparametric technology specifications (e.g., the choice between constant and variable returns to scale), it is common to choose the one having the best goodness-of-fit at the sample level for a given efficiency measure. But, since the efficiency measures are partially ordered (see above and Proposition 10), this goodness-of-fit approach may not be straightforward to implement when using it to select efficiency measures themselves.

4.2. Constant Elasticity of Substitution-Constant Elasticity of Transformation Model: LP Formulations

This subsection focuses on providing LPs to compute the generalized Färe-Lovell measure. This we do by considering the constant elasticity of substitution (CES)–constant elasticity of transformation (CET) model introduced by Färe et al. (1988) and extended in Boussemart et al. (2009). It consists in two parts: the output part is characterized by a CET formula, and the input part is characterized by a CES formula.

This CES-CET model can be seen as a generalization of the traditional linear models proposed by Charnes et al. (1978) and Banker et al. (1984). Moreover, it admits as a limiting case the multiplicative model proposed by Charnes et al. (1982), which is also discussed in Section 4.3.

These production models are useful in our context since they yield some tractable LPs. In particular, we first prove that the new generalized Färe-Lovell efficiency measure can be calculated by LPs under a suitable assumption regarding this model involving α-returns to scale.

For all positive real numbers r, we consider the map $\Phi_r : \mathbb{R}^d_+ \to \mathbb{R}^d_+$ defined as

$$\Phi_r(z) = \left(z_1^r, \ldots, z_d^r\right). \tag{37}$$

For all $r > 0$, this function is an isomorphism from $\mathbb{R}^d_+$ to itself, and its reciprocal is defined on $\mathbb{R}^d_+$ as

$$\Phi_r^{-1}(z) = \left(z_1^{1/r}, \ldots, z_d^{1/r}\right). \tag{38}$$

If $r < 0$, then $\Phi_r : \mathbb{R}^d_+ \to (\mathbb{R}_+ \cup \{\infty\})^d$ is the map defined as

$$\Phi_r(z) = \left(\phi_r(z_1), \ldots, \phi_r(z_d)\right). \tag{39}$$

where ϕ_r is the isomorphism defined in Equation (10). For all $r < 0$, this function is an isomorphism from $\mathbb{R}^d_+$ to $(\mathbb{R}_+ \cup \{\infty\})^d$, and its reciprocal is defined on $(\mathbb{R}_+ \cup \{\infty\})^d$ as

$$\Phi_r^{-1}(z) = \left(\phi^{-1}(z_1), \ldots, \phi^{-1}(z_d)\right). \tag{40}$$

For the sake of simplicity suppose that $A = \{(x_j, y_j) : j \in \mathcal{J}\} \subset \mathbb{R}^{n+m}_{++}$. In such a case, this function is an isomorphism from $\mathbb{R}^m_{++}$ to itself, and its reciprocal is defined on $\mathbb{R}^m_+$. Now, let us consider the following technology:

$$T_{p,q} = \left\{ (x, y) : x \geq \Phi_p^{-1}\left(\sum_{j \in \mathcal{J}} \theta_j \Phi_p(x_j)\right), \right. \\ \left. y \leq \Phi_q^{-1}\left(\sum_{j \in \mathcal{J}} \theta_j \Phi_q(y_j)\right), \theta \geq 0 \right\}, \tag{41}$$

where $p, q \in \mathbb{R}$ and $p \neq 0, q \neq 0$. This production model slightly differs from the one proposed by Färe et al. (1988) because of the suppression of the variable returns-to-scale constraint $\sum_{j\in\mathcal{J}} \theta_j = 1$. Notice that the topological limit of this technology is analyzed in Andriamasy et al. (2017). It is shown that one can retrieve as a special case the $\mathbb{B}$-convex production models proposed by Briec and Horvath (2009) and Briec and Liang (2011). The *DEA* technology corresponds to the case where $p = q = 1$.

Proposition 11. *Let $A = \{(x_j, y_j)\}_{j\in\mathcal{J}}$ be a set of $|\mathcal{J}|$ observed production vectors. Then, the production technology $T_{p,q}$ defined in (41) satisfies Assumptions 1–4.*

Now, we suppose that $p > 0$. We also assume that $q > 0$, though this condition might be reversed. It is easy to see that the new generalized Färe-Lovell input efficiency measure can be computed on $T_{p,q}$ using an LP. For the sake of simplicity, suppose that $x > 0$. We have the program

$$\begin{aligned}[G_p(x,y)]^p = \min \ & \frac{1}{n}\sum_{i=1}^{n} \beta_i{}^p \\ \text{s.t. } & \beta_i x_i \geq \left(\sum_{j\in\mathcal{J}} \theta_j x_{j,i}{}^p\right)^{\frac{1}{p}}, \quad i = 1,\ldots,n, \\ & y_k{}^q \leq \sum_{j\in\mathcal{J}} \theta_j y_{j,k}{}^q, \quad k = 1,\ldots,m, \\ & \beta_i \leq 1, \beta_i, \theta \geq 0. \end{aligned} \tag{42}$$

Since the map $p \mapsto a^p$ is increasing for all $p > 0$, we obtain

$$\begin{aligned}[G_p(x,y)]^p = \min \ & \frac{1}{n}\sum_{i=1}^{n} \beta_i{}^p \\ \text{s.t. } & \beta_i{}^p x_i{}^p \geq \sum_{j\in\mathcal{J}} \theta_j x_{j,i}{}^p \quad i = 1,\ldots,n, \\ & y_k{}^q \leq \sum_{j\in\mathcal{J}} \theta_j y_{j,k}{}^q, \quad k = 1,\ldots,m, \\ & \beta_i \leq 1, \beta_i, \theta \geq 0. \end{aligned} \tag{43}$$

Setting $(\beta_i)^p = u_i$ for all i yields an LP:

$$\begin{aligned}[G_p(x,y)]^p = \min \ & \frac{1}{n}\sum_{i=1}^{n} u_i \\ \text{s.t. } & u_i x_i{}^p \geq \sum_{j\in\mathcal{J}} \theta_j x_{j,i}{}^p \quad i = 1,\ldots,n, \\ & y_k{}^q \leq \sum_{j\in\mathcal{J}} \theta_j y_{j,k}{}^q, \quad k = 1,\ldots,m, \\ & u_i \leq 1, u_i, \theta \geq 0. \end{aligned} \tag{44}$$

We focus now on the case where $p < 0$. We also assume that $q < 0$, though such a condition can also be reversed. This is what we call the harmonic model. The term "harmonic" means the generalized sum is closely related to the harmonic mean. For the sake of simplicity, suppose that $x > 0$. We have the program

$$\begin{aligned}[G_p(x,y)]^p = \min \ & \frac{1}{n}\sum_{i=1}^{n} \beta_i{}^p \\ \text{s.t. } & \beta_i x_i \geq \left(\sum_{j\in\mathcal{J}} \theta_j x_{j,i}{}^p\right)^{\frac{1}{p}}, \quad i = 1,\ldots,n, \\ & y_k{}^q \leq \sum_{j\in\mathcal{J}} \theta_j y_{j,k}{}^q, \quad k = 1,\ldots,m, \\ & \beta_i \leq 1, \beta_i, \theta \geq 0. \end{aligned} \tag{45}$$

Since the map $p \mapsto a^p$ is decreasing for all $p < 0$, we obtain

$$\begin{aligned}[G_p(x,y)]^p = \max \ & \frac{1}{n}\sum_{i=1}^{n} \beta_i{}^p \\ \text{s.t. } & \beta_i{}^p x_i{}^p \leq \sum_{j\in\mathcal{J}} \theta_j x_{j,i}{}^p \quad i = 1,\ldots,n, \\ & y_k{}^q \geq \sum_{j\in\mathcal{J}} \theta_j y_{j,k}{}^q, \quad k = 1,\ldots,m, \\ & \beta_i \leq 1, \beta_i, \theta \geq 0. \end{aligned} \tag{46}$$

Setting $(\beta_i)^p = v_i$ for all i yields an LP:

$$\begin{aligned}[G_p(x,y)]^p = \max \ & \frac{1}{n}\sum_{i=1}^{n} v_i \\ \text{s.t. } & v_i x_i{}^p \leq \sum_{j\in\mathcal{J}} \theta_j x_{j,i}{}^p \quad i = 1,\ldots,n, \\ & y_k{}^q \geq \sum_{j\in\mathcal{J}} \theta_j y_{j,k}{}^q, \quad k = 1,\ldots,m, \\ & v_i \geq 1, \theta \geq 0. \end{aligned} \tag{47}$$

Notice that, in line with Chavas and Cox (1999), one could combine both a CES structure on the input side

Table 1. Numerical Example

Firms	Input 1	Input 2	Output
1	1	2	2
2	2	2	2
3	2	1	2
4	1	3	2
5	1	4	2
6	3	5/4	2
7	4	5/4	2

Table 2. Variation of the Efficiency Measures for Positive Values of p

Firms	$p=q=\frac{1}{4}$	$p=q=1$ Färe-Lovell DEA	$p=q=2$ Quadratic	$p=q=9$	$p=q=13$	Debreu-Farrell FDH
1	1	1	1	1	1	1
2	0.71779	0.75	0.790569	0.926075	0.948086	1
3	1	1	1	1	1	1
4	0.71779	0.833333	0.849837	0.92852	0.948451	1
5	0.71779	0.75	0.790569	0.926075	0.948086	1
6	0.5	0.733333	0.736357	0.755423	0.763693	0.8
7	0.318417	0.65	0.667083	0.74189	0.758591	0.8

and a harmonic structure on the output side by setting $p = 1 - r$ and $q = -r$, where r is a positive parameter.

4.3. Multiplicative Färe-Lovell Efficiency on Cobb-Douglass Model: LP Formulations

Based on Charnes et al. (1982) and Banker and Maindiratta (1986), we now consider the piecewise Cobb-Douglass (CD) model defined by

$$T_{CD} = \left\{(x,y) \in \mathbb{R}^{n+m} : x \geq \prod_{j\in\mathcal{J}} x_j^{\lambda_j}; y \leq \prod_{j=1\in\mathcal{J}} y_j^{\lambda_j}; \sum_{j\in\mathcal{J}} \lambda_j = 1; \lambda \geq 0\right\}. \quad (48)$$

In Andriamasy et al. (2017) it was shown that this production possibility set is the limit technology of $T_{p,p}$ when $p \longrightarrow 0$. For this model it is easy to show that the problem of computing the multiplicative Färe-Lovell measure can be converted to solving a simple LP. The program we need to solve for the multiplicative Färe-Lovell efficiency measure is

$$\begin{aligned} \mathrm{MFL}_\alpha(x,y) = \min \prod_{i=1,\ldots,n} \beta_i^{\alpha_i}, \\ \beta \odot x \geq \prod_{j\in\mathcal{J}} x_j^{\lambda_j}, \\ y \leq \prod_{j\in\mathcal{J}} y_j^{\lambda_j}, \\ \sum_{j\in\mathcal{J}} \lambda_j = 1, \lambda_j \geq 0. \end{aligned} \quad (49)$$

Applying a log-linear transformation to this program yields

$$\begin{aligned} \mathrm{MFL}_\alpha(x,y) = \min \sum_{i=1,\ldots,n} \alpha_i \ln\beta_i, \\ \ln x_i + \ln\beta_i \geq \sum_{j\in\mathcal{J}} \lambda_j \ln x_{j,i}, \quad i = 1,\ldots,n, \\ \ln y_k \leq \sum_{j\in\mathcal{J}} \lambda_j \ln y_{j,k}, \quad k = 1,\ldots,n, \\ \sum_{j\in\mathcal{J}} \lambda_j = 1, \lambda_j \geq 0. \end{aligned} \quad (50)$$

Setting $\ln\beta_i = \gamma_i$ for all i, one obtains an LP.

4.4. Numerical Example

The following numerical example in Table 1 is found in Färe et al. (1985, p. 76). Two inputs jointly produce a constant output level.

The values of the efficiency measures GFL_p under a production technology $T_{p,q}$ are listed in Table 2 and Table 3 for different values of p.

In Table 2, we consider positive values of p. The results are also compared with the Färe-Lovell efficiency scores ($p = 1$) computed using the traditional *DEA* model. In Table 2, the GFL_p is computed for each parameter p with respect to the technology $T_{p,p}$, where $q = p$. Since, in this example, the outputs are identical for each production vector, the value of q has no implication for the structure of the input set. In general, this is not the case. However, since the measure is input-oriented, one can derive an LP to compute the generalized Färe-Lovell efficiency scores. One can

Table 3. Variation of the Efficiency Measures for Negative Values of p

Firms	$p=q=-11$ Asymmetric Färe	$p=q=-11$	$p=q=-9$	$p=q=-1$ Harmonic	$p=q=-\frac{1}{4}$	$p=q=0$ Cobb-Douglas
1	1	1	1	1	1	1
2	0.5	0.603938	0.539913	0.666667	0.696583	0.707107
3	1	1	1	1	1	1
4	0.5	0.603938	0.717988	0.8	0.696583	0.707107
5	0.5	0.603938	0.539913	0.666667	0.696583	0.707107
6	0.25	0.342588	0.365738	0.645161	0.470996	0.5
7	0.2	0.285411	0.274305	0.526316	0.308056	0.316228

Table 4. Summary of Main Results

Value of p	Efficiency measure
$-\infty$	Asymmetric Färe
-1	Harmonic Färe-Lovell
0	Multiplicative Färe-Lovell
1	Färe-Lovell
2	Quadratic Färe-Lovell
$+\infty$	Debreu-Farrell

check that, when $p \to \infty$, then the Debreu-Farrell efficiency scores are close to the GFL_p score with $p = 13$. This is due to the fact that when $p \to \infty$ the technology topologically converges (in the Kuratowski-Painlevé sense) to the $\mathbb{B}$-convex model as shown by Andriamasy et al. (2017). In addition, since the data are constructed with a single identical output, the input set of the $\mathbb{B}$-convex model is identical to the *FDH* input set. This is also clearly the case when we compare the values for $p = 0$ (Cobb-Douglas case) and the model for $p = \frac{1}{4}$. These numerical results illustrate the limit properties established in Proposition 5.

Table 3 considers negative values of the parameter p. The procedure is similar, and one can see that, when $p \to -\infty$, the efficiency scores computed for each decision-making unit are close to those derived from the asymmetric Färe measure. This illustrates Proposition 8. Notice however, that in such a case we just consider a fixed value $p = -11$. Indeed, the limit technology obtained when $p \to -\infty$ is the inverse $\mathbb{B}$-convex model (see Andriamasy et al. 2017) for which there may not exist any efficient decision making in the input-oriented case.

Notice that the comparison rules provided in Proposition 10 (see Equation (32)) are in fact not broken when we reach negative values. For the needs of computation a specific technology $T_{p,p}$ is considered for any p. However, the comparison rules are established given a specific (fixed) production technology, independent of p. This is the reason why we observe an oscillating behavior when p is negative, since $T_{p,p}$ varies.

5. Concluding Comments

We have introduced a new generalized Färe-Lovell input efficiency measure. Thereafter, we have established that one can obtain the Debreu (1951) and Farrell (1957) input efficiency measure when p tends to infinity, the Färe and Lovell (1978) measure for $p = 1$, the asymmetric Färe measure when p tends to minus infinity, and finally the multiplicative Färe-Lovell measure when $p = 0$.

By way of summary, Table 4 lists how to obtain the main input efficiency measures as limiting cases for different values of the parameter p in the generalized input efficiency measure.

This paper has also outlined ways of implementation in the context of deterministic nonparametric technology specifications. Apart from a strong disposable flexible returns-to-scale technology, we have also developed two specifications that allow for LPs when solving for the generalized input Färe-Lovell measure and the multiplicative Färe-Lovell measures.

In brief, this generalized Färe-Lovell input efficiency measure offers a broad framework for empirical applications, and its encompassing nature provides a natural framework for testing the choice of efficiency measure. Obviously, the extension to alternative deterministic parametric specifications of technology (see Aigner and Chu 1968) is rather straightforward. Generalizations to alternative stochastic technology specifications (see, e.g., Kuosmanen and Johnson 2010) remain to be developed. Finally, note that in parametric stochastic specifications, nonradial efficiency measures are rarely if ever used. (Exceptions employing a unidimensional asymmetric Färe efficiency measure are Kumbhakar (1989) and Guarda et al. (2013).)

While in line with the economic literature this paper has concentrated on input efficiency measurement, these results can be easily transposed to output- or graph-oriented measurement orientations.[4] One could equally develop these results in a directional distance framework starting from Briec et al. (2011), who already defined the main input directional efficiency measures. As alluded to in the introduction, ideally one would like to have a way to derive the parameter p in the generalized Färe-Lovell measure in some endogenous way such that efficiency is measured relative to the subset of the technology where the majority of observations are situated. Finally, in addition to technical efficiency measurement, one can also focus on scale efficiency and the determination of global and local returns to scale. All of these developments are left for future work.

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Endnotes

[1] See also Briec et al. (2011) for such links in the context of input directional efficiency measures.

[2] The Nirenberg projection theorem is based on lecture notes from Nirenberg delivered in 1961 as found in, for example, Luenberger (1969, chapter 5).

[3] An alternative dual approach has been proposed in Kerstens and Vanden Eeckaut (1995): it was limited to a nonparametric technology context and focused on the Färe-Lovell, asymmetric Färe, and Zieschang (1984) measures. We ignore the latter efficiency measure, because it proceeds in two steps and it essentially combines the radial and the Färe-Lovell efficiency measures. However, paralleling

Zieschang (1984), one can easily construct a generalized Zieschang measure by combining together the generalized Färe-Lovell and the Debreu-Farrell measures.

[4] See, for example, the survey by Russell and Schworm (2011) on graph-oriented measures.

References

Aigner D, Chu S (1968) On estimating the industry production function. *Amer. Econom. Rev.* 58(4):826–839.

Andriamasy L, Briec W, Mussard S (2017) On some relations between several generalized convex DEA models. *Optimization* 66(4): 547–570.

Banker R, Maindiratta A (1986) Piecewise loglinear estimation of efficient production surfaces. *Management Sci.* 32(1): 126–135.

Banker R, Charnes A, Cooper W (1984) Some models for estimating technical and scale inefficiencies in Data Envelopment Analysis. *Management Sci.* 30(9):1078–1092.

Ben-Tal A (1977) On generalized means and generalized convex functions. *J. Optim. Theory Appl.* 21(1):1–13.

Boussemart JP, Briec W, Peypoch N, Tavéra C (2009) α-returns to scale and multi-output production technologies. *Eur. J. Oper. Res.* 197(1):332–339.

Briec W (2000) An extended Färe-Lovell technical efficiency measure. *Internat. J. Production Econom.* 65(2):191–199.

Briec W, Horvath C (2009) A $\mathbb{B}$-convex production model for evaluating performance of firms. *J. Math. Anal. Appl.* 355(1): 131–144.

Briec W, Liang Q (2011) On some semilattice structures for production technologies. *Eur. J. Oper. Res.* 215(3):740–749.

Briec W, Cavaignac L, Kerstens K (2011) Directional measurement of technical efficiency of production: An axiomatic approach. *Econom. Model.* 28(3):775–781.

Briec W, Kerstens K, Van de Woestyne I (2018) Hypercongestion in production correspondences: An empirical exploration. *Appl. Econom.* 50(27):2938–2956.

Charnes A, Cooper W, Rhodes E (1978) Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* 2(6):429–444.

Charnes A, Cooper W, Seiford L, Stutz J (1982) A multiplicative model for efficiency analysis. *Socio-Econom. Planning Sci.* 16(5): 223–224.

Chavas JP, Cox T (1999) A generalized distance function and the analysis of production efficiency. *Southern Econom. J.* 66(2): 294–318.

Debreu G (1951) The coefficient of resource utilization. *Econometrica* 19(3):273–292.

Färe R (1975) Efficiency and the production function. *Zeitschrift für Nationalökonomie* 35(3–4):317–324.

Färe R, Lovell C (1978) Measuring the technical efficiency of production. *J. Econom. Theory* 19(1):150–162.

Färe R, Grosskopf S, Lovell C (1985) *The Measurement of Efficiency of Production* (Kluwer, Boston).

Färe R, Grosskopf S, Njinkeu D (1988) On piecewise reference technologies. *Management Sci.* 34(12):1507–1511.

Färe R, Lovell C, Zieschang K (1983) Measuring the technical efficiency of multiple output production technologies. Eichhorn W, Neumann K, Shephard R, eds. *Quantitative Studies on Production and Prices* (Physica, Würzburg), 159–171.

Farrell M (1957) The measurement of productive efficiency. *J. Roy. Statist. Soc. Ser. A* 120(3):253–281.

Giannakas K, Tran K, Tzouvelekas V (2003) On the choice of functional form in stochastic frontier modeling. *Empirical Econom.* 28(1):75–100.

González E, Álvarez A (2001) From efficiency measurement to efficiency improvement: The choice of a relevant benchmark. *Eur. J. Oper. Res.* 133(3):512–520.

Guarda P, Rouabah A, Vardanyan M (2013) Identifying bank outputs and inputs with a directional technology distance function. *J. Production Anal.* 40(2):185–195.

Hackman S (2008) *Production Economics: Integrating the Microeconomic and Engineering Perspectives* (Springer, Berlin).

Hoffman KL (1981) A method for globally minimizing concave functions over convex sets. *Math. Programming* 20(1):22–32.

Jones D, Tamiz M (2010) *Practical Goal Programming* (Springer, New York).

Kerstens K, Vanden Eeckaut P (1995) Technical efficiency measures on DEA and FDH: A reconsideration of the axiomatic literature. CORE Discussion Paper 9513, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

Koopmans T (1951) Analysis of production as an efficient combination of activities. Koopmans T, ed. *Activity Analysis of Production and Allocation* (Yale University Press, New Haven, CT), 33–97.

Kopp R (1981a) The measurement of productive efficiency: A reconsideration. *Quart. J. Econom.* 96(3):477–503.

Kopp R (1981b) Measuring the technical efficiency of production: A comment. *J. Econom. Theory* 25(3):450–452.

Kopp R, Smith V (1980) Frontier production function estimates for steam electric generation: A comparative analysis. *Southern Econom. J.* 46(4):1049–1059.

Kumbhakar S (1989) Estimation of technical efficiency using flexible functional form and panel data. *J. Bus. Econom. Statist.* 7(2): 253–258.

Kuosmanen T, Johnson A (2010) Data Envelopment Analysis as nonparametric least-squares regression. *Oper. Res.* 58(1):149–160.

Lovell C, Pastor J (1995) Units invariant and translation invariant DEA models. *Oper. Res. Lett.* 18(3):147–151.

Luenberger D (1969) *Optimization by Vector Space Methods* (Wiley, New York).

Pastor J, Ruiz J, Sirvent I (1999) An enhanced DEA Russell graph efficiency measure. *Eur. J. Oper. Res.* 115(3):596–607.

Portela M, Thanassoulis E (2005) Profitability of a sample of Portuguese bank branches and its decomposition into technical and allocative components. *Eur. J. Oper. Res.* 162(3):850–866.

Portela M, Thanassoulis E (2007) Developing a decomposable measure of profit efficiency using DEA. *J. Oper. Res. Soc.* 58(4): 481–490.

Ruggiero J, Bretschneider S (1998) The weighted Russell measure of technical efficiency. *Eur. J. Oper. Res.* 108(2):438–451.

Russell R (1985) Measures of technical efficiency. *J. Econom. Theory* 35(1):109–126.

Russell R (1988) On the axiomatic approach to the measurement of technical efficiency. Eichhorn W, ed. *Measurement in Economics* (Physica, Heidelberg), 207–217.

Russell R (1990) Continuity of measures of technical efficiency. *J. Econom. Theory* 51(2):255–267.

Russell R, Schworm W (2009) Axiomatic foundations of efficiency measurement on data-generated technologies. *J. Production Anal.* 31(2):77–86.

Russell R, Schworm W (2011) Properties of inefficiency indexes on ⟨input, output⟩ space. *J. Production Anal.* 36(2):143–156.

Russell R, Schworm W (2018) Technological inefficiency indexes: A binary taxonomy and a generic theorem. *J. Production Anal.* 49(1):17–23.

Thanassoulis E, Dyson R (1992) Estimating preferred target input-output levels using Data Envelopment Analysis. *Eur. J. Oper. Res.* 56(1):80–97.

Tuy H, Thieu T, Thai N (1985) A conical algorithm for globally minimizing a concave function over a closed convex set. *Math. Oper. Res.* 10(3):498–514.

Zhu J (1996) Data Envelopment Analysis with preference structure. *J. Oper. Res. Soc.* 47(1):136–150.

Zieschang K (1984) An extended Farrell efficiency measure. *J. Econom. Theory* 33(2):387–396.

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