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Interfaces with Other Disciplines

Generalised commensurability properties of efficiency measures: Implications for productivity indicators[☆]

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We analyse the role of new weak and strong commensurability conditions on efficiency measures and especially on productivity measurement. If strong commensurability fails, then a productivity index (indicator) may exhibit a homogeneity bias yielding inconsistent and contradictory results. In particular, we show that the Luenberger productivity indicator is sensitive to proportional changes in the input-output quantities, while the Malmquist productivity index is not affected by such changes. This is due to the homogeneity degree of the directional distance function under constant returns to scale. In particular, the directional distance function only satisfies the weak commensurability axiom in general. However, if the directional distance function is a diagonally homogeneous function of the technology, then the directional distance function satisfies strong commensurability. This explains why the direction of an arithmetic mean of the observed data works well. Numerical examples and an empirical illustration are proposed. Under a translation homothetic technology, the Luenberger productivity indicator is not affected by any additive directional transformation of the observations.

1. Introduction

The purpose of this contribution is to point out some particular properties of a recent generalization of Shephard (1970) distance function, known as the directional distance function (DDF). Distance functions are employed in consumption and production theory. Luenberger (1992a,b) introduces the benefit function as a directional representation of preferences, which generalizes Shephard's (1970) input distance function defined in terms of the utility function. Luenberger (1995) introduces the shortage function as a transposition of the benefit function in a production context. Chambers, Chung and Färe (1996) relabel this same function as a DDF and since then it is commonly known by this name. The DDF generalizes existing Shephardian distance functions by accounting for both input reductions and output expansions and it is dual to the profit function (see Chambers, Chung, & Färe, 1998 for details). Furthermore, the DDF offers flexibility due to the variety of direction vectors it allows for (see, e.g., Chambers, Färe, & Grosskopf, 1996). Chambers, Chung and Färe (1996) analyze the benefit function as well as the DDF in detail and extend the composition rules of McFadden (1978) to these new concepts. However, it should be noted that there are alternative distance functions that the DDF fails to generalise: examples include the hyperbolic graph measure, the Hölder distance function for any norm, etc. (see, e.g., Russell & Schworm, 2011).

These Shephardian distance functions have been extensively used in the economic literature to measure productivity. Based upon Shephardian distance functions as general representations of technology, discrete-time Malmquist input- and output-oriented productivity indexes - introduced by Caves, Christensen, & Diewert (1982)- have been made empirically tractable by Färe, Grosskopf, Lindgren, & Roos (1995). Meanwhile, more general primal productivity indicators have been proposed. Chambers & Pope (1996) define a Luenberger productivity indicator (LPI) in terms of differences between DDFs (see also Chambers, 2002).[1]

Russell (1988) introduces an important property that any technical efficiency measure should satisfy: the commensurability con-

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[1] Note that traditional “indexes” denote productivity measures based on ratios while “indicators” use differences (see Diewert, 2005 for a detailed discussion).

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dition. This means that an efficiency measure should be invariant with respect to any change in the units of measurement. This condition is very natural and fundamental and most of the existing technical efficiency measures (or distance functions) satisfy it. This is the case for all the Shephardian measures, the Färe & Lovell (1978) measure, perhaps the first non-radial measure in the literature, as well as the Zieschang (1984) measure (see Russell & Schworm, 2009: footnote 8). Note that in the literature the commensurability property is also known under the name of unit(s) invariance.

Many of the new efficiency measures proposed in the literature involve some parameters in their definitions. This is the case of the measures proposed by Chambers, Chung and Färe (1996), Chavas & Cox (1999), Mehdiloozad, Sahoo, & Roshdi (2014), and Briec (1999), among others. Therefore, the notion of commensurability proposed by Russell (1988) must be modified to take into account these generalized structures. A first purpose of this contribution is then to generalise the commensurability notion to account for efficiency measures involving some parameters.

The second purpose of this contribution is to indicate the problems for measuring productivity when a measure fails to satisfy the commensurability property independently of the parameters it is depending on. For instance, the LPI that is related to the axiomatic properties of the directional distance function may yield some irrelevant and contradictory results depending on the direction that is chosen. Briec, Dervaux and Leleu (2003) show that the DDF satisfies a special version of the commensurability condition when the direction g is “pre-assigned”. Hence, the Russell (1988) commensurability condition cannot be applied to the DDF. To overcome this problem, we introduce a slight modification of the commensurability condition and we distinguish between two notions called weak and strong commensurability, respectively. Strong commensurability extends the original Russell (1988) commensurability notion to the case where distance functions involve specific parameters. It is shown that the directional distance function satisfies the weak commensurability but fails to satisfy strong commensurability. However, many of the existing efficiency measures do satisfy the strong version of the commensurability condition.

We apply the formalism suggested by Russell (1988) that associates an efficiency score to any pair of production vector and production technology. In general, a distance function (efficiency measure) is defined given a production technology. If the direction is a diagonally homogeneous function depending on the technology, then a slightly modified formulation of the DDF satisfies the strong commensurability condition. This explains why it is useful to consider the direction of an arithmetic mean of the observed data in empirical studies, as already suggested in Chambers, Färe and Grosskopf (1996: p. 185 and 190).

More importantly, under a constant returns to scale (CRS) assumption, an efficiency measure that does not satisfy the strong commensurability axiom cannot be homogeneous of degree 0. In such a case, one can show the existence of a productivity bias when a firm is proportionally re-scaled. In particular, the DDF is homogeneous of degree 1 under CRS. This property has some important implications concerning the LPI when the direction g is pre-assigned. In such a case, the LPI may yield some contradictory results, while the Malmquist productivity index provides very intuitive results in any case. Furthermore, it should be stressed that these properties are independent of the returns to scale structure of the production technology. If the technology satisfies a graph translation homotheticity property, then the LPI does not exhibit any bias when a firm is translated. Notice also that the fact that the DDF yields a radial expansion of a production vector is not problematic to evaluate technical efficiency, since the size of a firm may have some implication on its efficiency score.

Our empirical study shows that when the direction is proportional under a CRS assumption, then the results are consistent with those obtained in the Malmquist productivity index case. Some irrelevant and contradictory results appear when the direction is fixed independently of the technology. Interestingly, when the direction is fixed as the arithmetic mean of all the observed data, then the results are comparable to those obtained in the proportional case, with some minor differences. This confirms the interest of the latter specification as already proposed by Chambers, Färe and Grosskopf (1996).

To develop these arguments, this contribution is structured as follows. Section 2.1 develops the basic definitions of the technology and the various distance functions and efficiency measures. It provides two definitions of the commensurability property refining the axiom proposed by Russell (1988). Section 3 analyzes the implication of the commensurability condition on the consistency of productivity measurement. This we do by introducing a suitable notion of homogeneity bias. Section 4 provides a numerical example reporting some contradiction and irrelevant results. Section 5 proposes an empirical application comparing the result in the proportional and directional cases. We end with a concluding Section 6.

2. Technology and efficiency measures: definitions

2.1. Technology: definition and assumptions

A production technology describes how inputs $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_+$ are transformed into outputs $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$. The production possibility set T is the set of all feasible inputs and outputs vectors and it is defined as follows:

$$T = \left\{(x, y) \ \in \mathbb{R}^{m+n}_+ : \quad x \text{ can produce } y\right\}. \tag{2.1}$$

We suppose that the technology satisfies a series of usual assumptions or axioms:

(A.1) $(0,0) \in T,\ (0,y) \in T \Rightarrow y = 0$ (i.e., inaction, and no free lunch);
(A.2) For all $x \in \mathbb{R}^m_+$ the subset $A(x) = \{(u,y) \in T : u \leq x\}$ of dominating observations is bounded (i.e., infinite outputs cannot be obtained from a finite input vector);
(A.3) T is closed (i.e., closedness); and
(A.4) $\forall (x,y) \in T,\quad (u,v) \in \mathbb{R}^{m+n}_+$ and $(x,-y) \leq (u,-v) \Rightarrow (u,v) \in T$ (i.e., strong input and output disposability).
(A.5) $\forall (x,y) \in T$, and all $\lambda > 0$ $(\lambda x, \lambda y) \in T$ (i.e., CRS assumption).

The reader can consult Färe, Grosskopf, & Lovell (1994) for further comments on these axioms. Note that not all of the above axioms are needed to derive our main results.

2.2. Radial and directional efficiency measures

Distance functions fully characterise technology and for these reason have become standard tools for estimating efficiency and productivity relative to production frontiers. Let $\mathcal{T}$ be the class of all the production technologies satisfying the axioms $(A.1) - (A.4)$.

The radial input efficiency measure E_i is the inverse of the Shephard input distance function. It is the map $E^{\text{in}} : \mathbb{R}^{m+n}_+ \times \mathcal{T} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ defined as

$$E^{\text{in}}(x,y,T) = \inf_{\lambda}\{\lambda > 0 : (\lambda x, y) \in T\}. \tag{2.2}$$

The radial output efficiency measure $E^{\text{out}} : \mathbb{R}^{m+n}_+ \times \mathcal{T} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ searches for the maximum expansion of an output vector by a scalar θ to the production frontier, i.e.:

$$E^{\text{out}}(x,y,T) = \sup_{\theta}\{\theta > 0 : (x, \theta y) \in T\}. \tag{2.3}$$

The DDF is a map $\overrightarrow{D} : \mathbb{R}_+^{m+n} \times \mathbb{R}_+^{m+n} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{\infty, \infty\}$ defined by:

$$\overrightarrow{D}(x, y, h, k, T) = \sup_{\delta \in \mathbb{R}} \{\delta : (x - \delta h, y + \delta k) \in T\}. \tag{2.4}$$

It looks for a simultaneous input and output variation in the direction of a pre-assigned vector $g = (h, k) \in \mathbb{R}_+^{m+n}$ compatible with the technology (see Chambers, Färe & Grosskopf, 1996). The DDF is a special case of the shortage function (Luenberger, 1992b). It is also closely related to the translation function as developed in Blackorby & Donaldson (1980). Both functions measure the distance in a pre-assigned direction to the boundary of technology.

Färe, Grosskopf, & Margaritis (2008: p. 533-534) list a variety of choices for the direction vector. This question on the choice of direction vector has led to a rather substantial amount of literature proposing a variety of directions and also trying to determine some optimal type of direction vector in an endogenous way (see, for instance, Atkinson & Tsionas, 2016, Daraio & Simar, 2016, Layer, Johnson, Sickles, & Ferrier, 2020, Zofío, Pastor, & Aparicio, 2013 for representative examples). It is clear that the choice of direction vector affects the value of the DDF as well as its relative ranking: see, e.g., Kerstens, Mounir, & de Woestyne (2012) for an empirical illustration. Furthermore, Zofío et al. (2013) illustrate that when the direction vector is chosen to project inefficient firms towards profit maximizing benchmarks, then the traditional distinction between technical and allocative efficiency collapses: profit inefficiency can be categorized as either technical (when firms are situated in the interior of the technology) or allocative (when firms are situated on the frontier).

Finally, the proportional distance function (PDF) is introduced by Briec (1997). In the following we consider the Hadamard product defined for all $\gamma, z \in \mathbb{R}^d$ by

$$\gamma \odot z = (\gamma_1 z_1, \cdots, \gamma_d z_d).$$

This Hadamard product notation is useful to simplify the formulation of the PDF proposed by Briec (1997) who uses diagonal matrices. The PDF is the map $D^{\propto} : \mathbb{R}_+^{m+n} \times [0, 1]^{m+n} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ defined by

$$D^{\propto}(x, y, \alpha, \beta, T) = \sup_{\delta \in \mathbb{R}} \{\delta : (x - \delta\alpha \odot x, y + \delta\beta \odot y) \in T\}. \tag{2.5}$$

A special case corresponds to the situation where inputs and outputs are equiproportionaly modified. This implies that $\alpha = 1\!\!1_m$ and $\beta = 1\!\!1_n$. In such a case, we have:

$$\begin{aligned} D_T^{\propto}(x, y, T) &:= D_T^{\propto}(x, y, 1\!\!1_m, 1\!\!1_n) \\ &= \max\left\{\delta : \big((1 - \delta)x, (1 + \delta)y\big) \in T\right\}. \end{aligned} \tag{2.6}$$

It is generally stated in the literature that this PDF (2.5) is a special case of of the DDF (2.4) taking the direction $g = (-\alpha \odot x, \beta \odot y)$. Thus, we have:

$$\overrightarrow{D}(x, y, -\alpha \odot x, \beta \odot y, T) = D^{\propto}(x, y, \alpha, \beta, T). \tag{2.7}$$

However, note that in such a case g is not pre-assigned since it depends on x and y (see Russell & Schworm, 2011 : p. 146 for details).

In the following we establish under a CRS assumption that the DDF (2.4) is homogeneous of degree 1, while the PDF (2.5) is homogeneous of degree 0. The equiproportionate case ($\alpha = 1\!\!1_m$ and $\beta = 1\!\!1_n$) is established by Boussemart, Briec, Kerstens and Poutineau (2003) who show relationships between the radial and the proportional measures. This confirms that these distance functions are slightly different.

Briec, Dervaux and Leleu (2003: Prop. 1) establish that under a CRS assumption, the DDF is homogeneous of degree 1. Thus, if the technology satisfies a CRS assumption, then:

$$\overrightarrow{D}(\lambda x, \lambda y, g, T) = \lambda \overrightarrow{D}(x, y, g, T) \quad \forall \lambda \geq 0. \tag{2.8}$$

This result means that proportionally multiplying inputs and outputs by a scalar implies an equivalent proportional multiplication of the DDF. It is shown further that this property has some important implications for the LPI.

An overview of the axiomatic approach to input efficiency measures is found in Russell & Schworm (2009). A survey of efficiency measures in the graph of technology or in the full ⟨input, output⟩ space, like the DDFs and PDFs, is found in Russell & Schworm (2011) and in a more limited sense in Pastor & Aparicio (2010).

Note that in the remainder of this contribution, we use the simplified notations: $z = (x, y)$, $g = (h, k)$ and $\gamma = (\alpha, \beta)$.

2.3. Weak and Strong Commensurability of Efficiency Measures

This subsection revisits the commensurability condition proposed by Russell (1988: p. 21) in the input space only and by Russell & Schworm (2011) in the input-output or graph space.[2] In particular, we propose a new distinction between two notions of strong and weak commensurability. This distinctions is necessary since the introductions of efficiency measures depending on some parameters. This is obviously the case of both the DDFs and PDFs.

We first consider a set of variables $Z \subset \mathbb{R}^d$ an a set of parameters $\Theta \subset \mathbb{R}^{d'}$ where d and d' are two natural numbers. In the following, the Hadamard product is used to extend the commensurability concept in a proper way. Given any subset Z of $\mathbb{R}^d$ and any vector $c \in \mathbb{R}_{++}^d$, we denote $c \odot Z = \{c \odot z : z \in Z\}$. This notation is equivalent to the formulation proposed by Russell (1988: p. 212) who uses diagonal matrices. This formulation yields an equivalent formulation of the usual definition of commensurability.

Definition 2.1. Let Z be a subset of $\mathbb{R}^d$ and $\mathcal{S}$ be a collection of subsets of $\mathbb{R}^d$. Let $f : Z \times \mathcal{S} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$. We say that f satisfies the commensurability condition on Z if for all $c \in \mathbb{R}_{++}^d$, we have:

$$f(c \odot z, c \odot S) = f(z, S).$$

This definition is refined and extended as follows for a large class of maps involving some parameters.

Definition 2.2. Let Z be a subset of $\mathbb{R}^d$ and let $\mathcal{S}$ be a collection of subsets of $\mathbb{R}^d$. Let Θ be a subset of $\mathbb{R}^{d'}$. Let $f : Z \times \Theta \times \mathcal{S} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$. We say that f satisfies:

(a) A strong commensurability condition on Z and S if for all $c \in \mathbb{R}_{++}^d$, we have:

$$f(c \odot z, \theta, c \odot S) = f(z, \theta, S).$$

(b) A weak commensurability condition on Z and S if there exists a map $\xi : \mathbb{R}_{++}^d \mapsto \mathbb{R}_{++}^{d'}$ such that for all $c \in \mathbb{R}_{++}^d$:

$$f\big(c \odot z, \xi(c) \odot \theta, c \odot S\big) = f(z, \theta, S).$$

The map ξ is called a re-scaling function. It captures the fact that the parameters may be involved with the function f under any arbitrary algebraic form. Notice that strong commensurability implies weak commensurability when taking $\xi(c) = 1\!\!1_d$, for all c. However, in the remainder we focus on some cases where ξ is the identity map (such that $\xi(c) = c$ with $d = d'$). This implies that the re-scaling of the parameter θ is parallel to the one of the variable x. In many situations we consider the case where $Z = \mathbb{R}_+^{m+n}$ on which the distance functions are defined.

In the first case, one can see that the map f is invariant with respect to any change in the units of measurement and independent of the parameter θ. This definition extends the commensurability condition of Russell (1988) to the broad class of efficiency

[2] The survey of Russell & Schworm (2009) mentions the commensurability condition, but provides limited analysis.

measures involving additional parameters. This is not true in the second case, where solely the units of measurement of the parameter change.

Notice that this whole formalism can equivalently be formulated using definite positive diagonal matrices as it has been done in Russell (1988). However, the Hadamard product yields some simplifications in many statements. The next result shows that, given a map that satisfies a weak commensurability assumption, one can construct a commensurable map replacing the parameter with the point the function is evaluated at. This idea is implicitly used in Briec (1999) to construct a commensurable Hölder distance function.

Perhaps more importantly, defining a suitable diagonally homogeneous map, one can show that the strong commensurability of the PDF can be derived from the weak commensurability of the DDF.

Let E be a subset of $\mathbb{R}^d$. In the following we say that a map $\eta : E \longrightarrow E$ is multiplicative if for all $w, z \in E$, we have $\eta(w \odot z) = \eta(w) \odot \eta(z)$. A map $\kappa : E \longrightarrow E$ is diagonally homogeneous if for all $w, z \in E$, $\kappa(w \odot z) = w \odot \kappa(z)$. This property plays an important role in the analysis of commensurability. Note that we assume that the dimension of the vector space that contains the set of parameters is $d' = d$ and $Z = \mathbb{R}^d_+$.

Proposition 2.3. *Let $\mathcal{S}$ be a collection of subsets of $\mathbb{R}^d$ and let Θ be a subset of $\mathbb{R}^d$. Let $f : \mathbb{R}^d_+ \times \Theta \times \mathcal{S} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$. Suppose that f satisfies a weak commensurability condition on S and that $\xi : \mathbb{R}^d_{++} \mapsto \mathbb{R}^d_{++}$ is the associated rescaling function that is multiplicative. Let $\kappa : \mathbb{R}^d_+ \longrightarrow \mathbb{R}^d_+$ be a diagonally homogenous map.*

(a) Then, the map $g : \mathbb{R}^d_{++} \times \mathcal{S} \longrightarrow \mathbb{R}$ defined as:

$$g(z, S) = f\big(z, \xi \circ \kappa(z), S\big)$$

satisfies the strong commensurability condition for all $z \in \mathbb{R}^d_{++}$.

(b) Suppose that there exists a multiplicative extension $\tilde{\xi} : \mathbb{R}^d_+ \longrightarrow \mathbb{R}^d_+$ of ξ. Then, the map $\tilde{g} : \mathbb{R}^d_+ \times \mathcal{S} \longrightarrow \mathbb{R}$ defined as:

$$\tilde{g}(z, S) = f\big(z, \tilde{\xi} \circ \kappa(z), S\big)$$

satisfies the strong commensurability condition for all $z \in \mathbb{R}^d_+$.

The proof of this Proposition 2.3 as well as all other statements is found in Appendix A.

In the following, we show that the DDF satisfies the weak axiom of commensurability, but fails to satisfy the strong axiom. Both the radial efficiency measure and the PDF do satisfy the strong commensurability axiom. It is also shown that the PDF is homogeneous of degree 0. Recall that the DDF is homogeneous of degree 1.

In the next statement, we prove that the strong commensurability axiom implies homogeneity of degree 0 under a CRS assumption on technology.

Proposition 2.4. *Let $\mathcal{C}$ be the collection of all the conical subsets of $\mathbb{R}^d$. If $f : \mathbb{R}^d \times \Theta \times \mathcal{C} \longrightarrow \mathbb{R}$ satisfies the strong commensurability condition, then it is homogeneous of degree 0 in its first argument.*

Proposition 2.5. *The PDF (2.5) satisfies the strong commensurability axiom. The DDF (2.4) satisfies the weak commensurability axiom.*

Proposition 2.4 implies that a map that is not homogeneous of degree 0 under a CRS technology does not satisfy the strong commensurability condition. The second result of Proposition 2.5 is already found in Theorems 2 and 3 of Russell & Schworm (2011), in Briec, Dervaux and Leleu (2003), and in Pastor & Aparicio (2010). It is important to stress that the strong commensurability of the PDF can be derived from the weak commensurability of the DDF. For example, the map $\kappa : \gamma \odot z$ is diagonally homogeneous. Taking ξ as the identity map, that by definition is defined over $\mathbb{R}^n_+$, one can apply Proposition 2.3 to deduce the strong commensurability of the PDF using Eq. (2.7) that is obtained by replacing g with $\gamma \odot z$.

Notice that the Hölder distance function based upon a standard ℓ_p norm proposed in Briec (1999: p. 124) also fails to satisfy the strong commensurability axiom. Let us consider the norm:

$$(u, v) \mapsto \|(u, v)\|_{p,\gamma} = \left(\sum_{i\in[m]} \alpha_i |u_i|^p + \sum_{j\in[n]} \beta_j |v_j|^p\right)^{\frac{1}{p}}. \tag{2.9}$$

In the case where $p = \infty$, we have $\|(u, v)\|_{\infty,\gamma} = \max\{\max_{i\in[m]} \alpha_i |u_i|, \max_{j\in[n]} \beta_j |v_j|\}$. Briec (1999) defines the so-called Hölder distance function $D_{\|\cdot\|_p} : \mathbb{R}^{m+n}_+ \times \mathbb{R}^{m+n}_+ \times \mathcal{T} \longrightarrow \mathbb{R}$ defined for all $z \in T$ as

$$D_{\|\cdot\|_p}(z, \gamma, T) = \inf\left\{\|z - w\|_{p,\gamma} : w \in \partial_W(T)\right\}, \tag{2.10}$$

where $\partial_W(T) = \{(x, y) \in T : (u, -v) < (x, -y) \rightarrow (u, v) \notin T\}$ is the weakly efficient subset of the technology. Since for all $c = (a, b) \in \mathbb{R}^{m+n}_{++}$ we have $\partial_W(c \odot T) = c \odot \partial_W(T)$, it is easy to show that this Hölder distance function satisfies the weak commensurability using the re-scaling function

$$\xi(a, b) = \left(a_1^{-p}, \dots, a_m^{-p}, b_1^{-p}, \dots, b_n^{-p}\right), \tag{2.11}$$

where $c = (a, b)$. In the case where $(x, y) \in \mathbb{R}^{m+n}_{++}$, Briec (1999) shows that the commensurability can be obtained by setting $\alpha_i = x_i{}^{-p}$ and $\beta_j = y_j{}^{-p}$ respectively for all i, j. This means that we have replaced (α, β) with $\xi(x, y)$ and κ is the identity map. Therefore, such a property can be immediately derived from Proposition 2.3. In such a case, the map ξ cannot be extended to $\mathbb{R}^{m+n}_+$.

However, this result can be extended to the whole Euclidean vector space using a suitable restriction of the weak efficient subset. Notice that in the case of polyhedral norms ($p = 1, \infty$), the Hölder distance function is closely related to the DDFs and PDFs.

Proposition 2.6. *If the production technology satisfies a CRS assumption (A.5), then the PDF (2.5) is homogeneous of degree 0.*

The next Proposition 2.7 establishes a result which implies in Proposition 2.8 that the the DDF never satisfies the strong commensurability condition for technologies having a nonempty interior. Note that this assumption is often implicit for any production technology. In the following, for each subset E of $\mathbb{R}^d$, we denote by $\mathrm{int}(E)$ its interior.

Proposition 2.7. *Let us consider $c \in \mathbb{R}^{m+n}_{++}$ whose components are all identical and equal to $\lambda > 0$.*

(a) If $\lambda > 1$, then for all $z \in T$:

$$\vec{D}(c \odot z, g, c \odot T) \geq \lambda \vec{D}(z, g, T).$$

If $z \in \mathrm{int}(T)$, then $\vec{D}(c \odot z, g, c \odot T) > \vec{D}(z, g, T)$.

(b) If $\lambda \in]0, 1[$, then for all $z \in T$:

$$\vec{D}(c \odot z, g, c \odot T) \leq \lambda \vec{D}(z, g, T).$$

If $z \in \mathrm{int}(T)$, then $\vec{D}(c \odot z, g, c \odot T) < \vec{D}(z, g, T)$.

(c) If T satisfies a CRS assumption (A.5), then:

$$\vec{D}(c \odot z, g, c \odot T) = \lambda \vec{D}(z, g, T).$$

Moreover, for all $z \in \mathrm{int}(T)$, if $\lambda \neq 1$, then $\vec{D}(c \odot z, g, c \odot T) \neq \vec{D}(z, g, T)$.

In particular, Proposition 2.7 means that any homogeneous expansion (contraction) of the units of measurement implies an expansion (contraction) of the DDF. Consequently, the DDF does not satisfy the strong commensurability axiom, since one can always find a technology in $\mathcal{T}$ which violates the strong commensurability condition, although the DDF satisfies weak commensurability (as shown in Proposition 2.5).

Proposition 2.8. *The DDF (2.4) does not satisfy the strong commensurability axiom.*

This result is perfectly general and it challenges the widespread use of the DDF as an efficiency measure. We illustrate this lack of commensurability in a LPI context.

In the following, we suggest a slight change in the traditional definition of the DDF. Let $g : \mathcal{T} \longrightarrow \mathbb{R}_+^{m+n}$ be a vector valued map defined as: $g : T \mapsto \big(h(T), k(T)\big)$. Let $\mathcal{F}$ be the set of all the maps defined from $\mathcal{T}$ to $\mathbb{R}_+^{m+n}$. The map $\overrightarrow{D}^\sharp : \mathbb{R}_+^{m+n} \times \mathcal{F} \times \mathcal{T}$ defined as:

$$\overrightarrow{D}^\sharp(x, y, g, T) = \sup\left\{\delta : \big(x - \delta h(T), y + \delta k(T)\big) \in T\right\} \tag{2.12}$$

is called the adjusted directional distance function (ADDF). Equivalently, we have:

$$\overrightarrow{D}^\sharp(x, y, g, T) = \overrightarrow{D}(x, y, g(T), T). \tag{2.13}$$

Notice that this definition does not involve any fixed parameter: g is just assumed to be a functional defined over $\mathcal{T}$. We say that $g : \mathbb{R}_+^{m+n} \longrightarrow \mathbb{R}_+^{m+n}$ is diagonally homogeneous over $\mathcal{T}$, if for all $c \in \mathbb{R}_{++}^d$, we have $g(c \odot T) = c \odot g(T)$. In the following, it is shown that one can provide a sufficient condition for the strong commensurability of $\overrightarrow{D}^\sharp(x, y, g, T)$.

Proposition 2.9. *If g is diagonally homogeneous, then the ADDF (2.12) is strongly commensurable.*

It is not clear that the diagonal homogeneity of g is a necessary condition for strong commensurability. For example, the PDF is strongly commensurable though the direction is not fixed. This condition, however, provides a technical argument to one of the specifications proposed by Chambers, Färe and Grosskopf (1996) in a nonparametric context.

Let us denote $\mathcal{P} = \langle \mathbb{R}_+^{m+n} \rangle$ the set of all the finite parts of $\mathbb{R}_+^{m+n}$. Let Λ be the set of all the diagonally homogeneous set-valued maps $\widetilde{T} : \mathcal{P} \rightrightarrows \mathcal{T}$. Let $\widetilde{T}(\mathcal{P}) = \{\widetilde{T}(A) : A \in \mathcal{P}\}$ and let $\mathcal{T}_\Lambda = \left\{\widetilde{T}(\mathcal{P}) : \widetilde{T} \in \Lambda\right\}$ be the set of all the production technologies indexed in Λ and $\mathcal{P}$. $\mathcal{T}_\Lambda$ encompasses as a special case a large class of non-parametric production models. Suppose that $A = \{(x_1, y_1), (x_2, y_2), ..., (x_\ell, y_\ell)\}$ is a set of ℓ observed production vectors. For all $A \in \mathcal{P}$, let $Cc(A)$ and $Co(A)$ respectively denote the conical hull and the convex hull of A and let $K = \mathbb{R}_+^m \times \mathbb{R}_-^n$ be the free disposal cone. If $\widetilde{T}_C$ is the set-valued map defined by $\widetilde{T}_C(A) = (Cc(A) + K) \cap \mathbb{R}_+^{m+n}$, then $\widetilde{T}_C(A)$ corresponds to a CRS specification (see, e.g., Briec & Lemaire, 1999). If $\widetilde{T}_V$ is the map defined by $\widetilde{T}_V(A) = (\mathbb{R}_+^m \times \{0\}) \cup (Co(A) + K) \cap \mathbb{R}_+^{m+n}$, then $\widetilde{T}_V(A)$ corresponds to a variable returns to scale model, completed with the inaction point (0,0) (to satisfy A.1). This procedure is not limited to convex nonparametric models: for instance, a basic Free Disposal Hull model is obtained from the application $\widetilde{T}_F$ defined as $\widetilde{T}_F(A) = \{(0, 0)\} \cup \big(A + K\big) \cap \mathbb{R}_+^{m+n}$.

Taking the direction

$$g = \left(\frac{1}{\ell}\sum_{k\in[\ell]} x_k, \frac{1}{\ell}\sum_{k\in[\ell]} y_k\right), \tag{2.14}$$

the DDF is independent of any change in the units of measurements. This property can be related to Proposition 2.9. Actually, note that two distinct data sets may yield the same technology. To overcome such a problem, let us introduce the equivalence relation $A \sim A' \iff \widetilde{T}(A) = \widetilde{T}(A')$ and let $\tilde{\mathcal{P}} = \mathcal{P} \backslash \sim$ the set of the corresponding equivalence classes, that is the quotient set. Let $\Psi : \widetilde{T}(\mathcal{P}) \longrightarrow \tilde{\mathcal{P}}$ which associates to any $T \in \widetilde{T}(\mathcal{P})$ some $\tilde{A} \in \mathcal{P}$ such that $\widetilde{T}(A) = T$ for all $A \in \tilde{A}$. By construction, for all $c \in \mathbb{R}_{++}^{m+n}$, we have $\widetilde{T}(c \odot A) = c \odot \widetilde{T}(A)$ and this implies that $\Psi\big(c \odot \widetilde{T}(A)\big) = \Psi\big(\widetilde{T}(c \odot A)\big) = c \odot \tilde{A} = c \odot \Psi\big(\widetilde{T}(A)\big)$. It follows that $\Psi(c \odot T) = c \odot \Psi(T)$. Now, let us consider the map $m^\sharp : \tilde{P} \longrightarrow \mathbb{R}_+^{m+n}$ that associates to any equivalence class the arithmetic mean of some arbitrary element of this equivalence class. Namely, $m^\sharp(\tilde{A}) = \frac{1}{|A^\sharp|}\sum_{a\in A^\sharp} a$ where for any $\tilde{A}$, $A^\sharp$ is an arbitrary element of $\tilde{A}$. We retrieve the approach proposed by Chambers, Färe and Grosskopf (1996) and Färe, Grosskopf and Margaritis (2008) by defining the function $g : \widetilde{T}(\mathcal{P}) \longrightarrow \mathbb{R}_+^n$ as:

$$g(T) = m^\sharp\big(\Psi(T)\big). \tag{2.15}$$

Since $\Psi(c \odot T) = c \odot \Psi(T)$ and $m^\sharp(c \odot \Psi(T)) = c \odot m^\sharp(\Psi(T))$, we deduce that $g(c \odot T) = c \odot g(T)$. Notice that in such a case the direction depends on the sample of units. Therefore, the DDF is not translation invariant, as already mentioned in Aparicio, Pastor, & Vidal (2016). Suppose that A is a subset of $\mathbb{R}_{++}^{m+n}$, one could assume that the direction is a generalized mean of the observed production vectors with for all $(i, j) \in [m] \times [n]$

$$h_i = \left(\sum_{k\in[\ell]} x_{k,i}{}^{\alpha_i}\right)^{\frac{1}{\alpha_i}} \text{ and } k_j = \left(\sum_{k\in[\ell]} y_{k,j}{}^{\beta_j}\right)^{\frac{1}{\beta_j}}, \tag{2.16}$$

and $\alpha_i, \beta_j \neq 0$ for all i, j. For example, if $\alpha_i, \beta_j \longrightarrow \infty$ and $\alpha_i, \beta_j \longrightarrow -\infty$, then we have the limit case:

$$g = \left(\bigvee_{k\in[\ell]} x_k, \bigvee_{k\in[\ell]} y_k\right) \text{ and } g = \left(\bigwedge_{k\in[\ell]} x_k, \bigwedge_{k\in[\ell]} y_k\right), \tag{2.17}$$

where $\vee$ and $\wedge$ are the sup and inf lattice operator, respectively. Note that these results do no contradict Proposition 2.8. In Propositions 2.8 and 2.4, the parameters (direction) are assumed to be independent of T. Layer, Johnson, Sickles and Ferrier (2020) study how the shape of the nonparametric frontier estimation may impact the optimal direction. Along this line, they propose an analysis showing that setting the median of the variables as a direction tends to outperform the choice of other directions. In such a case, we have:

$$h_i = \text{med}\{x_{k,i} : k \in [\ell]\} \text{ and } k_j = \text{med}\{y_{k,j} : k \in [\ell]\}, \tag{2.18}$$

where med stands for the median. Obviously, the median direction also respects the commensurability condition of the ADDF.

Our research has focused here only on the Hölder distance function, the PDF and DDF, and the ADDF. It may be worthwhile exploring in future work to which extent other graph-oriented efficiency measures analysed in Russell & Schworm (2011) and in Pastor & Aparicio (2010) comply with this generalised commensurability definition. Having established that the Hölder distance function and the DDF only satisfy weak commensurability, it is time to explore the empirical consequences for productivity measurement. Since the DDF is far more popular in empirical research than the Hölder distance function, the next section focuses on how weak commensurability may affect the empirical results of the very popular LPI.

3. Productivity indices and indicators: implications of commensurability

Recently, quite a bit of attention has been devoted to so-called theoretical productivity indices (see Russell, 2018). A theoretical productivity index is defined on the assumption that the technology is known and non-stochastic, but unspecified and thus most often approximated by a nonparametric specification of technology using some form of efficiency measure. The foundational concepts are on the one hand the Malmquist productivity index (Caves, Christensen and Diewert, 1982) and on the other hand the Hicks-Moorsteen productivity index (Bjurek, 1996). While the Malmquist productivity index is fundamentally a measure of the shift of the production frontier, the Hicks-Moorsteen productivity index is a ratio of an aggregate output index over an aggregate input index.

Thus, the Malmquist productivity index measures local technical change (i.e., the local shifts in the production frontier), while the Hicks-Moorsteen productivity index has a Total factor Productivity (TFP) interpretation. Kerstens & de Woestyne (2014) empirically illustrate that the Malmquist productivity index offers a poor approximation to the Hicks-Moorsteen TFP index in terms of the resulting distributions and that these problems persist under CRS as well as under variable returns to scale (VRS).

Chambers, Färe and Grosskopf (1996) introduce the LPI as a difference-based indicator of DDFs (see Chambers, 2002). This generalizes the Malmquist productivity index that is most often either input- or output-oriented. Briec & Kerstens (2004) define a Luenberger-Hicks-Moorsteen TFP indicator using input- or output-oriented DDFs. LPIs and Luenberger-Hicks-Moorsteen productivity indicators are also empirically quite different under CRS as well as under VRS (see Kerstens, Shen, & de Woestyne, 2018). We now formally define the output-oriented Malmquist productivity index and the LPI that we need in our empirical analysis.

3.1. Productivity indices and indicators: definitions

At each time period let us denote T_t the production technology at the time period t and suppose that T_t satisfies axioms $(A.1)-(A.4)$. Productivity indices and indicators aim to evaluate productivity changes between discrete time periods and can be decomposed to analyse the origins in the productivity changes.

The Malmquist productivity index can be based on the radial output measure (2.3). In particular, Caves, Christensen and Diewert (1982) suggest using a geometric mean between a period t Malmquist productivity index $M_t^{\text{out}}(z_t, z_{t+1}, T_t)$:

$$M_t^{\text{out}}(z_t, z_{t+1}, T_t) = \frac{E^{\text{out}}(z_t, T_t)}{E^{\text{out}}(z_{t+1}, T_t)}, \tag{3.1}$$

and a period $t+1$ Malmquist productivity index $M_{t+1}^{\text{out}}(z_t, z_{t+1}, T_{t+1})$:

$$M_{t+1}^{\text{out}}(z_t, z_{t+1}, T_{t+1}) = \frac{E^{\text{out}}(z_t, T_{t+1})}{E^{\text{out}}(z_{t+1}, T_{t+1})}. \tag{3.2}$$

Similarly, Färe, Grosskopf, Lindgren and Roos (1995) define the output-oriented Malmquist productivity index as the geometric mean of (3.1) and (3.2) as follows:

$$M^{\text{out}}(z_t, z_{t+1}, T_t, T_{t+1}) = \left[\frac{E^{\text{out}}(z_{t+1}, T_t)}{E^{\text{out}}(z_t, T_t)} \frac{E^{\text{out}}(z_{t+1}, T_{t+1})}{E^{\text{out}}(z_t, T_{t+1})}\right]^{1/2}. \tag{3.3}$$

This productivity index allows to analyze productivity changes between different periods and it can be multiplicatively decomposed into efficiency changes (EC) and technological changes (TC):

$$EC = \frac{E^{\text{out}}(x_t, y_t, T_t)}{E^{\text{out}}(x_{t+1}, y_{t+1}, T_{t+1})} \quad \text{and} \quad TC = \left(\frac{E^{\text{out}}(z_{t+1}, T_{t+1})}{E^{\text{out}}(z_{t+1}, T_t)} \frac{E^{\text{out}}(z_t, T_{t+1})}{E^{\text{out}}(z_t, T_t)}\right)^{\frac{1}{2}}, \tag{3.4}$$

where EC represents the variation in efficiency between two periods and concerns the relative efficiency in the management of input and output quantities over time, while TC captures technological changes (i.e., productivity growth not explained by changes in input and output quantities).

The LPI based on the DDF (2.4) is defined as follows:

$$L(z_t, z_{t+1}, g, T_t, T_{t+1}) = \frac{1}{2}\Big[\vec{D}(z_t, g, T_{t+1}) - \vec{D}(z_{t+1}, g, T_{t+1}) + \vec{D}(z_t, g, T_t) - \vec{D}(z_{t+1}, g, T_t)\Big]. \tag{3.5}$$

This LPI can be additively decomposed into efficiency changes (EC) and technological changes (TC):

$$EC_t = \vec{D}(z_t, g, T_t) - \vec{D}(z_{t+1}, g, T_{t+1}) \tag{3.6}$$

and

$$TC_t = \frac{1}{2}\Big[\vec{D}(z_{t+1}, g, T_{t+1}) - \vec{D}(z_{t+1}, g, T_t) + \vec{D}(z_t, g, T_{t+1}) - \vec{D}(z_t, g, T_t)\Big], \tag{3.7}$$

where the interpretation follows the one provided for the Malmquist productivity index (3.3).

Paralleling this definition, Boussemart, Briec, Kerstens and Poutineau (2003) define a proportional Luenberger indicator based on the PDF (2.5) as:

$$L^{\propto}(z_t, z_{t+1}, \gamma) = \frac{1}{2}\Big[D^{\propto}(z_t, \gamma, T_{t+1}) - D^{\propto}(z_{t+1}, \gamma, T_{t+1}) + D(z_t, \gamma, T_t) - D^{\propto}(z_{t+1}, \gamma, T_t)\Big]. \tag{3.8}$$

The decomposition defined in (3.6) and (3.7) is applicable to this proportional case as well. Note that recently Pastor, Lovell, & Aparicio (2020) manage to transgress the distinction between technology and TFP indices outlined above. These authors define a new graph oriented inefficiency measure based on the PDF under CRS and use it to define a new Malmquist productivity index that has a TFP interpretation.

Early discussions by Ray & Desli (1997) and Lovell (2003), among others, have led to refinements to the basic decomposition of the output-oriented Malmquist productivity index (3.4) to account for the role of returns to scale. This has led to lively discussions about the correct (tautological) decomposition of the Malmquist productivity index. Early and somewhat dated surveys on this multiplicative decomposition of the Malmquist productivity index are found in Lovell (2003) and Zofío (2007). These discussions somewhat straightforwardly transpose to the LPI that has an additive structure.

However, Proposition 2.8 is perfectly general and, in particular, it is independent of any returns to scale assumption. Therefore, all decompositions of the LPI are potentially affected by the lack of strong commensurability of the DDF.

Notice that while the LPI does not require a CRS specification of the technologies, the large majority of empirical applications still imposes such a restrictive assumption.[3] Therefore, given space limitations this contributions limits itself to documenting the impact of the lack of strong commensurability of the LPI to the CRS case in both the numerical examples in Section 4 and the empirical illustration in Section 5.

3.2. Productivity indices and indicators: homogeneity bias

This subsection analyzes the impact of the commensurability condition on productivity measurement. We define a suitable notion of homogeneity bias for productivity indices and indicators. We also establish a relation between such a notion and the commensurability of the efficiency measure upon which a productivity index or indicator is based.

Definition 3.1. Let Θ be a subset of $\mathbb{R}^d$. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$. Let $T_t, T_{t+1} \in \mathcal{T}$. For all, $(z_t, z_{t+1}, \theta) \in T_t \times T_{t+1} \times \Theta$ and all $\lambda > 0$:

$$B_t(z_t, z_{t+1}, \phi, \theta, \lambda) = \phi(z_t, z_{t+1}, \theta, T_t, T_{t+1}) - \phi(\lambda z_t, z_{t+1}, \theta, T_t, T_{t+1})$$

is called the homogeneity bias of ϕ in period t;

[3] We provide some qualitative evidence for this claim. A Google Scholar search on 22 January 2022 yields about 979 results for the search term "Luenberger productivity indicator". This same search term in conjunction with the search term "constant returns to scale" obtains 422 hits, while this same search term in conjunction with the search term "variable returns to scale" leads to 383 results.

$$B_{t+1}(z_t, z_{t+1}, \phi, \theta, \lambda) = \phi(z_t, z_{t+1}, \theta, T_t, T_{t+1}) - \phi(z_t, \lambda z_{t+1}, \theta, T_t, T_{t+1})$$

is called the homogeneity bias of ϕ in period $t+1$.

The homogeneity bias measures the change of a productivity index or indicator when a firm is proportionally re-scaled at the time periods t and $t+1$. Since productivity is essentially based upon the ratio between the outputs and the inputs involved in the production process, one could expect that a productivity index or indicator should be invariant with respect to such a re-scaling when the technology satisfies a CRS assumption.

In the case of the LPI based on the DDF (2.4) the homogeneity bias in t is then defined as:

$$B_t(z_t, z_{t+1}, L, g, \lambda) = L(z_t, z_{t+1}, g, T_t, T_{t+1}) - L(\lambda z_t, z_{t+1}, g, T_t, T_{t+1}), \tag{3.9}$$

and the same homogeneity bias at the time period $t+1$ is defined as:

$$B_{t+1}(z_t, z_{t+1}, L, g, \lambda) = L(z_t, z_{t+1}, g, T_t, T_{t+1}) - L(z_t, \lambda z_{t+1}, g, T_t, T_{t+1}). \tag{3.10}$$

In the case of the proportional LPI based on the PDF (2.5) we have the homogeneity bias in t:

$$B_t(z_t, z_{t+1}, L^{\propto}, \gamma) = L^{\propto}(z_t, z_{t+1}, \gamma, T_t, T_{t+1}) - L^{\propto}(\lambda z_t, z_{t+1}, \gamma, T_t, T_{t+1}), \tag{3.11}$$

and the homogeneity bias in $t+1$:

$$B_{t+1}(z_t, z_{t+1}, L^{\propto}, \gamma) = L^{\propto}(z_t, z_{t+1}, \gamma, T_t, T_{t+1}) - L^{\propto}(z_t, \lambda z_{t+1}, \gamma, T_t, T_{t+1}). \tag{3.12}$$

Finally, the output-oriented Malmquist productivity index is independent of any parameter. Hence, for all $\theta \in \mathbb{R}^d$, we have the homogeneity bias in t:

$$B_t(z_t, z_{t+1}, M^{\text{out}}, \theta, \lambda) = M^{\text{out}}(z_t, z_{t+1}, T_t, T_{t+1}) - M^{\text{out}}(\lambda z_t, z_{t+1} T_t, T_{t+1}), \tag{3.13}$$

and the homogeneity bias in $t+1$:

$$B_{t+1}(z_t, z_{t+1}, M^{\text{out}}, \theta, \lambda) = M^{\text{out}}(z_t, z_{t+1}, T_t, T_{t+1}) - M^{\text{out}}(z_t, \lambda z_{t+1} T_t, T_{t+1}). \tag{3.14}$$

The next result shows that given any efficiency measure satisfying the strong commensurability axiom, the corresponding productivity index or indicator has a null homogeneity bias.

Proposition 3.2. *Let Θ be a subset of $\mathbb{R}^d$. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$. Let $T_t, T_{t+1} \in \mathcal{T}$ and assume that T_t and T_{t+1} satisfy a CRS assumption. If ϕ satisfies the strong commensurability condition, then for all $(z_t, z_{t+1}, \theta) \in T_t \times T_{t+1} \times \Theta$ and all $\lambda > 0$,*

$$B_t(z_t, z_{t+1}, \phi, \theta, \lambda) = B_{t+1}(z_t, z_{t+1}, \phi, \theta, \lambda) = 0.$$

In the following, let:

$$B_{t,t+1}(z_t, z_{t+1}, \phi) = B_t(z_t, z_{t+1}, \phi) + B_{t+1}(z_t, z_{t+1}, \phi), \tag{3.15}$$

denote the sum of the homogeneity bias in time period t and in time period $t+1$. The next result shows that the homogeneity bias of the proportional LPI (3.8) and Malmquist productivity index (3.3) are null, though this is not the case for the LPI (3.5) based on the DDF for which an explicit form of the bias can be provided.

Corollary 3.3. *Suppose that at each time period T_t and T_{t+1} satisfy $(A.1)-(A.4)$ and a CRS assumption $(A.5)$. For all $(z_t, z_{t+1}) \in T_t \times T_{t+1}$ we have:*

(*a*) $B_t(z_t, z_{+1}, M^{\text{out}}, \theta, \lambda) = B_{t+1}(z_t, z_{+1}, M^{\text{out}}, \theta, \lambda) = 0$;

(*b*) $B_t(z_t, z_{+1}, L^{\propto}, \gamma, \lambda) = B_{t+1}(z_t, z_{+1}, L^{\propto}, \alpha, \beta, \lambda) = 0$;

(*c*) *We have the identities:*

$$B_t(z_t, z_{t+1}, g, \lambda) = \frac{1-\lambda}{2}\left[\overrightarrow{D}(z_t, g, T_{t+1}) + \overrightarrow{D}(z_t, g, T_t)\right];$$

$$B_{t+1}(z_t, z_{t+1}, g, \lambda) = \frac{\lambda - 1}{2}\left[\overrightarrow{D}(z_{t+1}, g, T_{t+1}) + \overrightarrow{D}(z_{t+1}, g, T_t)\right]; \text{ and}$$

$$B_{t,t+1}(z_t, z_{t+1}, g, \lambda) = \frac{1-\lambda}{2} L(z_t, z_{t+1}, g, T_t, T_{t+1}).$$

Under a CRS assumption on technology, the Malmquist productivity index and the proportional LPI are not affected by a proportional modification of one of the observations. However, this is not true in the case of the LPI based on the DDF. Remark that Chambers, Färe and Grosskopf (1996): p. 184) in their seminal article do impose a CRS assumption on technology.

3.3. Translation homothetic bias

In this subsection, it is shown that the things are very different when one assumes a graph translation homothetic property of the technology. First, notice that it is difficult to define the commensurability axiom from an additive viewpoint. This is due to the fact that the key axioms $(A.1)-(A.4)$ are not preserved via a translation of the technology. However, it is interesting to analyze the impact of the graph translation homotheticity on the structure of the LPI (3.5).

We point to the fact that if the technology is graph translation homothetic, then the LPI with a fixed direction does not suffer from the shortcomings due to its additive structure. A production technology T is translation homothetic in the direction of g if for all $z \in T$ and all $\delta \in \mathbb{R}$ such that $z + \delta g \in \mathbb{R}_+^{m+n}$, we have $z + \delta g \in T$. It was shown by Briec & Kerstens (2004) that under an assumption of graph translation homotheticity:

$$D(z + \delta g, g, T) = D(z, g, T). \tag{3.16}$$

This means that the DDF is translation invariant.

Paralleling our earlier definition we define the translation homothetic bias as follows.

Definition 3.4. Let Θ be a subset of $\mathbb{R}^d$. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$. Let $T_t, T_{t+1} \in \mathcal{T}$. For all, $(z_t, z_{t+1}, \theta) \in T_t \times T_{t+1} \times \Theta$ and all $\lambda > 0$:

$$TB_t(z_t, z_{t+1}, \phi, \theta, \delta) = \phi(z_t, z_{t+1}, \theta, T_t, T_{t+1}) - \phi(z_t + \delta g, z_{t+1}, \theta, T_t, T_{t+1})$$

is called the translation homothetic bias of ϕ in period t;

$$TB_{t+1}(z_t, z_{t+1}, \phi, \theta, \delta) = \phi(z_t, z_{t+1}, \theta, T_t, T_{t+1}) - \phi(z_t, z_{t+1} + \delta g, \theta, T_t, T_{t+1})$$

is called the translation homothetic bias of ϕ in period $t+1$.

In the case of the LPI (3.5) the translation homothetic bias in t is then defined as:

$$TB_t(z_t, z_{t+1}, L, g, \delta) = L(z_t, z_{t+1}, g, T_t, T_{t+1}) - L(z_t + \delta g, z_{t+1}, g, T_t, T_{t+1}); \tag{3.17}$$

and the translation homothetic bias at the time period $t+1$ is then:

$$TB_{t+1}(z_t, z_{t+1}, L, g, \delta) = L(z_t, z_{t+1}, g, T_t, T_{t+1}) - L(z_t, z_{t+1} + \delta g, g, T_t, T_{t+1}). \tag{3.18}$$

It follows that if the production technology is graph translation homothetic at both the time periods t and $t+1$, then:

$$TB_t(z_t, z_{t+1}, L, g, \delta) = TB_{t+1}(z_t, z_{t+1}, L, g, \delta) = 0. \tag{3.19}$$

This means that the translation homotheticity bias is zero.

4. Numerical examples

In the following we compare the output-oriented Malmquist productivity index and the LPI. To do so we introduce a numerical example and we show that the LPI can yield inconsistent results because of the structure of the DDF under a CRS assumption.

4.1. Output-oriented measures

We suppose that the technology is two-dimensional and that $T_0 = \{(x, y) : y \le x\}$ and $T_1 = \{(x, y) : y \le 2x\}$, which implies a CRS assumption at each time period. Moreover, we assume that: $z_0 = (x_0, y_0) = (1, \frac{4}{5})$ and $z_1 = (x_1, y_1) = (1, \frac{5}{4})$.

Let us compute the radial output-oriented efficiency measure at each time period:

(i) $E^{\text{out}}(z_1, T_0) = \sup\{\theta : (1, \theta\frac{5}{4}) \in T_0\} = \sup\{\theta : \theta\frac{5}{4} \le 1\}$. Clearly, we have $\frac{5}{4}\theta^\star = 1$ and $E^{\text{out}}(z_1, T_1) = \theta^\star = \frac{4}{5}$;
(ii) $E^{\text{out}}(z_0, T_0) = \sup\{\theta : \theta\frac{4}{5} \le 1\}$, hence $E^{\text{out}}(z_0, T_0) = \theta^\star = \frac{5}{4}$;
(iii) $E^{\text{out}}(z_1, T_1) = \sup\{\theta : \theta\frac{5}{4} \le 2\}$. Clearly, we have $\frac{5}{4}\theta^\star = 2$ and $E^{\text{out}}(z_1, T_1) = \theta^\star = \frac{8}{5}$;
(iv) $E^{\text{out}}(z_0, T_1) = \sup\{\theta : \theta\frac{4}{5} \le 2\}$, hence $E^{\text{out}}(z_1, T_1) = \theta^\star = \frac{5}{2}$.

Inserting these results leads to the following output-oriented Malmquist productivity index (3.3):

$$M^{\text{out}}(z_0, z_1, T_0, T_1) = \left(\frac{5}{4}\cdot\frac{5}{4}\cdot\frac{5}{8}\cdot\frac{2}{5}\right)^{\frac{1}{2}} = 1.56. \tag{4.1}$$

This result indicates a productivity gain between $t = 0$ and $t = 1$, since indeed the Malmquist productivity index is > 1.

Now we suppose that $\lambda = 10$. It follows that we consider the production vector at $t = 1$ defined as:

$$z'_1 = 10(x_1, y_1) = \left(10, \frac{25}{2}\right).$$

Although in the first and the second case the observation do not use the same level of inputs and outputs, these observations have the same efficiency scores. Thus, the productivity index should yield the same result. This is indeed the case for the Malmquist productivity index, since it is invariant with respect to a proportional change of the second observation.

$$E(z'_1, T_0) = \frac{4}{5}, E(z_0, T_0) = \frac{5}{4}, E(z'_1, T_1) = \frac{8}{5}, E(z_0, T_1) = \frac{5}{2}.$$

Hence, inserting these results we also obtain:

$$M^{\text{out}}(z_0, 10z_1, T_0, T_1) = \left(\frac{5}{4}\cdot\frac{5}{4}\cdot\frac{5}{2}\cdot\frac{5}{8}\right)^{\frac{1}{2}} = 1.56.$$

Thus, a proportional multiplication of z_1 by 10 does not affect the output-oriented Malmquist productivity index. This is normal because the productivity does not change.

But, for the LPI (3.5) such proportional change in input and output quantities does affect the indicator, thereby introducing a bias. Recall that as in the Malmquist productivity index case, the production vectors are $z_0 = (1, \frac{4}{5})$ and $z_1 = (1, \frac{5}{4})$. Let us now consider the LPI with the direction of $g = (0, 1)$. This is an output-oriented LPI which allows to be compared with the output-oriented Malmquist productivity index:

(i) $\vec{D}(x_0, y_t, 0, 1, T_1) = \sup\{\delta : (1, \frac{4}{5} + \delta) \in T_1\}$ which implies that $\frac{4}{5} + \delta^\star = 2$ and $\vec{D}(x_0, y_0, 0, 1, T_1) = \delta^\star = \frac{6}{5}$;
(ii) $\vec{D}(x_1, y_1, 0, 1, T_1) = \sup\{\delta : (1, \frac{5}{4} + \delta) \in T_1\}$. Hence, $\vec{D}(x_1, y_1, 0, 1, T_1) = \frac{3}{4}$;
(iii) $\vec{D}(x_0, y_0, 0, 1, T_0) = \sup\{\delta : (1, \frac{4}{5} + \delta) \in T_t\}$. Hence, $\frac{4}{5} + \delta = 1$ and $\vec{D}(x_0, y_0, 0, 1, T_0) = \frac{1}{5}$;
(iv) $\vec{D}(x_1, y_1, 0, 1, T_0) = \sup\{\delta : (1, \frac{5}{4} + \delta) \in T_0\}$. Hence, $\vec{D}(x_1, y_1, 0, 1, T_0) = -\frac{1}{4}$.

Inserting these results leads to the following output-oriented LPI:

$$L^{\text{out}}(z_0, z_1, 0, 1, T_0, T_1) = \frac{1}{2}\left[\frac{6}{5} - \frac{3}{4} + \frac{1}{5} + \frac{1}{4}\right] = \frac{1}{2}\cdot\frac{9}{10} = 0.45. \tag{4.2}$$

Since this LPI is larger than zero, this suggests a productivity gain between periods $t = 0$ and $t = 1$.

Now in the second case, the observation is again characterized by the following conditions: $z_0 = (x_0, y_0) = (1, \frac{4}{5})$ and $z'_1 = 10(x_1, y_1) = (10, \frac{25}{2})$.

Again, we compute the output-oriented DDF at each time period:

(i) $\vec{D}(x_0, y_0, 0, 1, T_1) = \sup\{\delta : (1, \frac{4}{5} + \delta) \in T_1\}$ which implies that $\frac{4}{5} + \delta = 2$ and $\delta = \frac{6}{5}$;
(ii) $\vec{D}(x_1, y_1, 0, 1, T_1) = \sup\{\delta : (10, \frac{25}{2} + \delta) \in T_{t+1}\}$ which implies that $\frac{25}{2} + \delta = 20$ so $\delta = \frac{15}{2}$;
(iii) $\vec{D}(x_0, y_0, 0, 1, T_0) = \sup\{\delta : (1, \frac{4}{5} + \delta) \in T_t\}$ which implies that $\frac{4}{5} + \delta = 1$ and therefore $\delta = \frac{1}{5}$;
(iv) $\vec{D}(x_1, y_1, 0, 1, T_0) = \sup\{\delta : (10, \frac{25}{2} + \delta) \in T_t\}$ Thus, $\frac{25}{2} + \delta = 10$ so $\delta = \frac{-5}{2}$.

Collecting again these results leads now to the following output-oriented LPI result:

$$L(z_0, 10z_1, g, T_0, T_1) = \frac{1}{2}\left[\frac{6}{5} - \frac{15}{2} + \frac{1}{5} + \frac{5}{2}\right] = \frac{1}{2}\cdot\left(\frac{-18}{5}\right) = -1.8. \tag{4.3}$$

Remark that the output-oriented LPI is now negative (−1.8) while it was initially positive (0.45). Thus, the LPI initially suggests a productivity gain, while it now indicates a productivity loss. However, this is a contradiction: in both cases the observation should have the same productivity. Therefore, the LPI is very sensitive to proportional changes in quantities and it does not allow to estimate changes in efficiency.

4.2. Graph-oriented measures

Fig. 1 illustrates the idea behind the homogeneity bias. When a production vector is proportionally expanded, then the DDF is increasing. Hence, the LPI may be significantly modified.

Consider the production vectors $z_0 = (1, \frac{3}{4})$ and $z_1 = (1, \frac{5}{4})$.

Let us compute the LPI based on the PDF (3.8) as introduced by Boussemart, Briec, Kerstens and Poutineau (2003). We consider the case where $\alpha = \mathbb{1}_m$ and $\beta = \mathbb{1}_n$. At each time periods t, s we have

$$D^\propto(x_t, y_t, T_s) = \max_\delta\{\delta : ((1 - \delta)x_t, (1 + \delta)y_t) \in T_s\}. \tag{4.4}$$

Under a CRS assumption, we have the relation:

$$D^\propto(x_t, y_t, T_s) = \frac{E^{\text{out}}(x_t, y_t, T_s) - 1}{E^{\text{out}}(x_t, y_t, T_s) + 1}. \tag{4.5}$$

Boussemart, Briec, Kerstens and Poutineau (2003) define the LPI based on the PDF as follows:

$$\begin{aligned} &L^\propto(x_t, y_t, x_{t+1}, y_{t+1}, T_t, T_t + 1) \\ &= \frac{1}{2}\Big[D^\propto(x_t, y_t, T_t) - D^\propto(x_{t+1}, y_{t+1}, T_t) \\ &+ D^\propto(x_t, y_t, T_{t+1}) - D^\propto(x_{t+1}, y_{t+1}, T_{t+1})\Big]. \end{aligned} \tag{4.6}$$

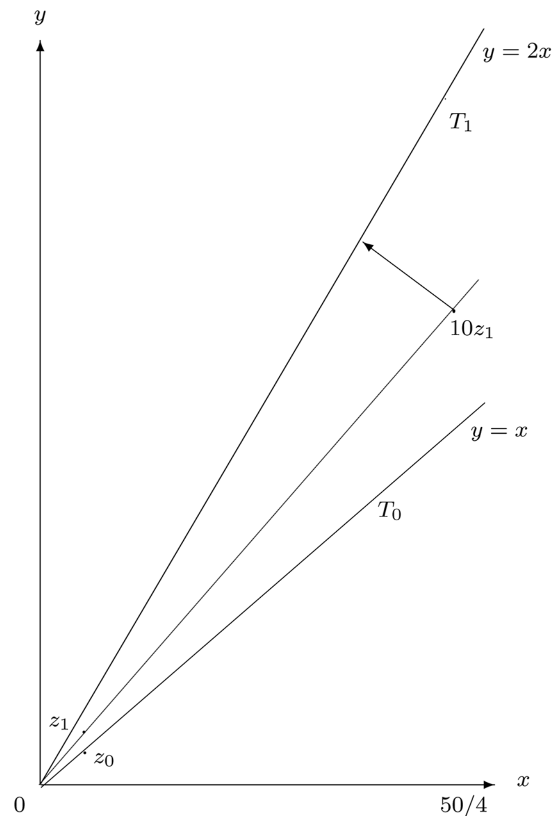

Fig. 1. Homogeneity Bias.

Since the PDF is homogenous of degree 0, we obviously have for all $\lambda > 0$:

$$L^{\propto}(x_t, y_t, x_{t+1}, y_{t+1}, T_t, T_{t+1}) = L^{\propto}(x_t, y_t, \lambda x_{t+1}, \lambda y_{t+1}, T_t, T_{t+1}). \tag{4.7}$$

Moreover, from Boussemart, Briec, Kerstens and Poutineau (2003), we also have under a CRS assumption, the second order approximation:

$$L^{\propto}(x_t, y_t, x_{t+1}, y_{t+1}, T_t, T_{t+1}) \approx \frac{1}{2}\ln\left(M^{\text{out}}(x_t, y_t, x_{t+1}, y_{t+1}, T_t, T_{t+1})\right). \tag{4.8}$$

Assuming that $z_0 = (1, \frac{4}{5})$, $z_1 = (1, \frac{5}{4})$, one can compute the PDFs at each time period as follows:

(i) $D^{\propto}(x_0, y_0, T_1) = \max\{\delta : (1-\delta, \frac{4}{5} + \frac{4}{5}\delta) \in T_1\}$. Hence, we should have $\frac{4}{5} + \frac{4}{5}\delta = 2(1-\delta)$ and $\delta = \frac{3}{7}$;
(ii) $D^{\propto}(x_1, y_1, T_1) = \max\{\delta : (1-\delta, \frac{5}{4} + \frac{5}{4}\delta) \in T_1\}$ so $\frac{5}{4} + \frac{5}{4}\delta = 2(1-\delta)$ and $\delta = \frac{3}{13}$;
(iii) $D^{\propto}(x_0, y_0, T_0) = \max\{\delta : (1-\delta, \frac{4}{5} + \frac{4}{5}\delta) \in T_0\}$. Thus, $\frac{4}{5} + \frac{4}{5}\delta = 1-\delta$ and $\delta = \frac{1}{7}$;
(iv) $D^{\propto}(x_1, y_1, T_0) = \max\{\delta : (1-\delta, \frac{5}{4} + \delta) \in T_0\}$. Hence, we deduce $\delta = -\frac{1}{9}$

Inserting these results yields the following proportional LPI:

$$L^{\propto}(z_0, z_1, T_0, T_1) = \frac{1}{2}\left[\frac{5}{11} - \frac{3}{13} + \frac{1}{7} + \frac{1}{9}\right] = 0.238. \tag{4.9}$$

Suppose now that $z_1 = (10, \frac{25}{2})$, since the PDF is homogeneous of degree 0, we have:

$$L^{\propto}(z_0, z_1, T_0, T_1) = L^{\propto}(z_0, 10z_1) = 0.238. \tag{4.10}$$

Therefore, the productivity change is the same. The results are parallel to those obtained using the output-oriented Malmquist productivity index.

Let us now compute the LPI based on the DDF (3.5) as follows:

(i) $\overrightarrow{D}(x_0, y_0, 1, 1, T_1) = \sup\{\delta : (1-\delta, \frac{3}{4} + \delta) \in T_1\}$. Thus so $\frac{3}{4} + \delta = 2(1-\delta)$ and $\delta = \frac{5}{12}$;
(ii) $\overrightarrow{D}(x_1, y_1, 1, 1, T_1) = \sup\{\delta : (1-\delta, \frac{5}{4} + \delta) \in T_1\}$ thus $\delta = \frac{3}{12}$;
(iii) $\overrightarrow{D}(x_0, y_0, 1, 1, T_0) = \sup\{\delta : (1-\delta, \frac{3}{4} + \delta) \in T_0\}$ so $\frac{3}{4} + \delta = 1-\delta$ and $\delta = \frac{1}{8}$;
(iv) $\overrightarrow{D}(x_1, y_1, 1, 1, T_0) = \sup\{\delta : (1-\delta, \frac{5}{4} + \delta) \in T_0\}$, thus $\delta = -\frac{1}{8}$.

Inserting these results into the LPI yields:

$$L(z_0, z_1, g, T_0, T_1) = \frac{1}{2}\left[\frac{5}{12} - \frac{3}{12} + \frac{1}{8} + \frac{1}{8}\right] = \frac{1}{2}\left(\frac{5}{12}\right) = 0.21. \tag{4.11}$$

Thus, this LPI being larger than > 0 suggests a productivity gain between periods $t = 0$ and $t = 1$.

Now in the second case the production vectors become $z_0 = (x_0, y_0) = (1, \frac{3}{4})$ and $z'_1 = 10(x_1, y_1) = (10, \frac{25}{2})$.

The DDFs in each time period are now:

(i) $\overrightarrow{D}(x_0, y_0, h, k, T_1) = \frac{5}{12}$;
(ii) $\overrightarrow{D}(x_1, y_1, h, k, T_1) = \sup\{\delta : (10-\delta, \frac{25}{2} + \delta) \in T^1\}$ so $\frac{25}{2} + \delta = 2(10-\delta)$ and $\delta = \frac{15}{6}$;
(iii) $\overrightarrow{D}(x_0, y_0, h, k, T_0) = \frac{1}{8}$;
(iv) $\overrightarrow{D}(x_1, y_1, h, k, T_0) = \sup\{\delta : (10-\delta, \frac{25}{2} + \delta) \in T^0\}$ so $\delta = -\frac{5}{4}$.

Collecting these results leads to the following LPI:

$$L(z_0, 10z_1, g, T_0, T_1) = \frac{1}{2}\left[\frac{5}{12} - \frac{15}{6} + \frac{1}{8} + \frac{5}{4}\right] = \frac{1}{2}\left(-\frac{17}{24}\right) = -0.35. \tag{4.12}$$

Since the indicator is now negative, it suggests a productivity loss between periods $t = 0$ and $t = 1$.

Again, one can remark contradictory results between these two cases. The LPI based on the DDF fails to measure productivity changes properly. This is due to the homogeneity degree of the DDF.

These numerical results are summarized in Table 1.

5. Empirical illustration

As an empirical illustration, we propose to focus on the schooling productivity of European countries using the PISA-OECD and Eurostat data. Indeed, PISA (Programme for International Student Assessment) is an OECD program that aims to evaluate the performances of educational systems of OECD member countries. Since 2000 and every three years, surveys are conducted to evaluate 15-year-olds' ability to use their reading, mathematics, and science knowledge in 36 OECD member countries and partner countries. In parallel, Eurostat collects and harmonizes published data from national statistics institutes of European Union countries for various themes like education.

To analyze schooling productivity, we consider as outputs the PISA reading scores, mathematics scores, and science scores in 2018 and 2009 of 15-year-olds' pupils to measure schooling productivity over almost one decade. Following Agasisti, Munda, & Hippe (2019), as inputs we select three types of resources: student/teacher ratio, government expenditure per student, and total public expenditure on education as percent of GDP. Furthermore, we distinguish those inputs for primary and secondary education levels and consider those resources during the schooling of pupils,

Table 1
Malmquist index and luenberger indicator: numerical examples.

	Case 1	Productivity	Case 2	Productivity
Output case	$z_t = (1, \frac{4}{5})$		$z_t = (1, \frac{4}{5})$	
	$z_{t+1} = (1, \frac{5}{4})$		$z_{t+1} = (10, \frac{50}{4})$	
Malmquist	$M^o = 1.56 > 1$	+	$M^o = 1.56 > 1$	+
Luenberger	$L = 0.45 > 0$	+	$L = -1.8 < 0$	-
Graph case	$z_t = (1, \frac{3}{4})$		$z_t = (1, \frac{3}{4})$	
	$z_{t+1} = (1, \frac{5}{4})$		$z_{t+1} = (10, \frac{50}{4})$	
Proportional	$L^{\propto} = 0.238 > 0$	+	$L^{\propto} = 0.238 > 0$	+
Luenberger	$L = 0.21 > 0$	+	$L = -0.35 < 0$	-

Table 2
Description of inputs and outputs.

Variable	Label	Time Period 0	Time Period 1
Output 1	Reading scores	2009	2018
Output 2	Mathematic scores	2009	2018
Output 3	Science scores	2009	2018
Input 1	student/teacher ratio (inverse) for primary education	2003 (except: Estonia 2001)	2012 (except: Greece 2013)
Input 2	student/teacher ratio (inverse) for secondary education	2007	2016 (except: Norway 2017)
Input 3	Government expenditure per student (based on FTE) for primary education (PPS)	2003 (except: Estonia 2005; Greece 2005; Hungary 2004)	2012 (except: Belgium 2011; Norway 2011)
Input 4	Government expenditure per student (based on FTE) for secondary education (PPS)	2007 (except: Hungary 2006)	2016
Input 5	Total public expenditure on primary, lower and upper secondary education as % of GDP for primary education	2003	2012 (except: Slovakia 2011)
Input 6	Total public expenditure on primary, lower and upper secondary education as % of GDP for secondary education	2007 (except: Greece 2005)	2016

Table 3
Productivity scores and ranking.

Country	Malmquist	Rank	LPI PDF	Rank	LPI DDF	Rank	LPI Mean	Rank
Italy	1,117	1	0,055	1	1,114	1	0,059	1
Sweden	1,111	2	0,052	2	0,492	3	0,055	2
Estonia	1,081	3	0,039	3	0,675	2	0,037	3
Austria	1,039	4	0,019	4	-0,278	8	0,027	4
Portugal	1,001	5	0,000	5	-0,419	11	-0,006	6
Netherlands	0,998	6	-0,001	6	-0,045	4	-0,003	5
UK	0,977	7	-0,012	7	-0,127	5	-0,011	7
France	0,959	8	-0,021	9	-0,213	7	-0,018	8
Norway	0,959	9	-0,020	8	-0,721	15	-0,024	10
Hungary	0,934	10	-0,034	10	-0,378	9	-0,020	9
Germany	0,932	11	-0,035	11	-0,399	10	-0,032	11
Greece	0,923	12	-0,040	12	-0,503	12	-0,033	12
Belgium	0,914	13	-0,043	13	-0,696	14	-0,053	14
Czechia	0,889	14	-0,058	14	-0,174	6	-0,034	13
Slovenia	0,875	15	-0,067	15	-1,270	20	-0,054	15
Latvia	0,845	16	-0,084	16	-0,836	17	-0,066	17
Poland	0,828	17	-0,093	17	-0,528	13	-0,062	16
Slovakia	0,786	18	-0,119	18	-0,958	18	-0,080	18
Finland	0,781	19	-0,123	19	-1,551	21	-0,130	21
Lithuania	0,715	20	-0,162	20	-0,754	16	-0,093	19
Bulgaria	0,686	21	-0,181	21	-1,053	19	-0,099	20
Average	0,921		-0,044		-0,411		-0,031	

i.e., for primary education in 2003 and 2012 so theoretically when pupils are 9-year-olds' and for secondary education in 2007 and 2016 so theoretically when pupils are 13-years-olds'. The reader can consult Table 2 for more details on these data. A sample of 21 European Union countries is collected. The original data can be found in Table B.1 in Appendix B.

We compute on these data four productivity indices and indicators: (i) the output-oriented Malmquist index (3.3), (ii) the input-oriented LPI based on the PDF (3.8), (iii) the input-oriented LPI based on DDF (3.5) with input direction: (0.01, 0.01, 1000, 1000, 0.1, 0.1), and (iv) the input-oriented LPI based on DDF (3.5) with as input direction the means in the sample (0.073, 0.096, 4609.34, 6211.84, 1.254, 2.039). The results and the rankings obtained for each index and indicator are presented in Table 3. In the top row, these four productivity indices and indicators are labeled "Malmquist", "LPI PDF", "LPI DDF" and "LPI Mean", respectively. The mathematical programming problems for these indices and indicators are found in Appendix C.

Note that in this empirical illustration we opt for input-oriented LPIs rather than graph-oriented ones. This methodological choice avoids any complications due to infeasibilities (see Briec & Kerstens, 2009a) and due to the need for positivity constraints on the projection of the outputs (see Briec & Kerstens, 2009b).

Our results show similar sign interpretation and ranking for the Malmquist productivity index and for the proportional LPI. But, for the LPI based on the DDF, the results are different. Indeed, the ranking is seriously modified. Some countries are better ranked with the directional LPI (Czechia (+8); Lithuania (+4), Poland (+4)), while some other countries are worse ranked (Norway (-7), Portugal (-6), Austria (-4), Slovenia (-4)). We also notice that the sign interpretation of the productivity indices and indicators is even inverted for Austria. Indeed, the Malmquist index and the proportional LPI highlight that Austria has increased its schooling productivity between 2009 and 2018 by 3.8 %, whereas the directional LPI reveals a productivity decrease for this same period of time. The countries are of different size and the choice of a preassigned direction that is independent from the observed data has a strong impact on the results. This also explains the difference between the efficiency scores and the evaluation of productivity and it confirms that strong commensurability is intimately linked to the robustness of the results.

Finally, using inputs means as direction for the directional LPI somewhat limits this issue. This confirms the idea that the choice of a direction as the mean of the observed data also yields relevant results. Therefore, the strong commensurability, inherited from the diagonal homogeneity of the direction, has a significant impact on the evaluation of productivity changes as shown in Proposition 2.4. The results indeed become closer to the Malmquist productivity index and the proportional LPI. This confirms that the choice of the direction as an arithmetic means of the observed production vectors yields more relevant results.

While Layer, Johnson, Sickles and Ferrier (2020) investigate the impact of measurement error on a stochastic DDF estimated using convex nonparametric least squares in a Monte Carlo simulation framework, their key findings are similar. First, directions close to the average orthogonal direction to the true function perform best. Second, with noisy data selecting a direction that matches the noise direction of the data generating process improves estimator performance.

6. Conclusion

We have refined the notion of commensurability and have shown that it plays a crucial role in the measurement of efficiency and productivity. An efficiency measure or distance function that is not strongly commensurable is not homogeneous of degree 0 under a CRS assumption. Therefore, it may yield wrong evaluations when empirically measuring efficiency and productivity.

This contribution has verified in detail some numerical examples and an empirical illustration in which it is shown that the LPI based upon the DDF may not be a relevant productivity indicator under any returns to scale assumption. The simplest alternative to avoid these problems is to employ the LPI based upon the PDF.

An avenue for future work is to explore in more detail to which extent other graph-oriented efficiency measures analysed in Russell & Schworm (2011) and in Pastor & Aparicio (2010) comply with this generalised commensurability definition and satisfy the property of strong commensurability. In addition to the Hölder distance function and the DDF, it may well be that other graph-oriented efficiency measures only satisfy weak commensurability and therefore may provide dubious productivity measures. Another open issue worthwhile exploring is to check to which extent overall efficiency concepts (e.g., based on the cost, revenue, or profit function) as well as the allocative efficiency notions comply with the commensurability conditions.[4] Furthermore, it could be useful to also empirically investigate how the Luenberger-Hicks-Moorsteen indicator is affected in a similar way like the LPI in terms of the choice of directions for the input- and output oriented DDF composing it. Finally, our numerical examples and empirical illustration could be complemented by some Monte Carlo analysis (similar to Layer et al. (2020)).

[4] In the case of cost efficiency Aparicio, Pastor, & Zofío (2017) show that the DDF does not correctly encompasses the allocative efficiency component of the Shephardian approach.

Declaration of Competing Interest

None

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2022.03.037.

References

Agasisti, T., Munda, G., & Hippe, R. (2019). Measuring the efficiency of european education systems by combining data envelopment analysis and multiple-criteria evaluation. *Journal of Productivity Analysis, 51*(2-3), 105–124.

Aparicio, J., Pastor, J. T., & Vidal, F. (2016). The directional distance function and the translation invariance property. *Omega, 58*, 1–3.

Aparicio, J., Pastor, J. T., & Zofío, J. L. (2017). Can Farrell's allocative efficiency be generalized by the directional distance function approach? *European Journal of Operational Research, 257*(1), 345–351.

Atkinson, S. E., & Tsionas, M. G. (2016). Directional distance functions: Optimal endogenous directions. *Journal of Econometrics, 190*(2), 301–314.

Bjurek, H. (1996). The Malmquist total factor productivity index. *Scandinavian Journal of Economics, 98*(2), 303–313.

Blackorby, C., & Donaldson, D. (1980). A theoretical treatment of indices of absolute inequality. *International Economic Review, 21*(1), 107–136.

Boussemart, J.-P., Briec, W., Kerstens, K., & Poutineau, J. C. (2003). Luenberger and Malmquist productivity indices: Theoretical comparisons and empirical illustration. *Bulletin of Economic Research, 55*(4), 391–405.

Briec, W. (1997). A graph type extension of Farrell technical efficiency measure. *Journal of Productivity Analysis, 8*(1), 95–110.

Briec, W. (1999). Hölder distance function and measurement of technical efficiency. *Journal of Productivity Analysis, 11*(2), 111–131.

Briec, W., Dervaux, B., & Leleu, H. (2003). Aggregation of directional distance functions and industrial efficiency. *Journal of Economics, 79*(3), 237–261.

Briec, W., & Kerstens, K. (2004). A Luenberger-Hicks-Moorsteen productivity indicator: Its relation to the Hicks-Moorsteen productivity index and the Luenberger productivity indicator. *Economic Theory, 23*(4), 925–939.

Briec, W., & Kerstens, K. (2009a). Infeasibilities and directional distance functions: With application to the determinateness of the Luenberger productivity indicator. *Journal of Optimization Theory and Applications, 141*(1), 55–73.

Briec, W., & Kerstens, K. (2009b). The Luenberger productivity indicator: An economic specification leading to infeasibilities. *Economic Modelling, 26*(3), 597–600.

Briec, W., & Lemaire, B. (1999). Technical efficiency and distance to a reverse convex set. *European Journal of Operational Research, 114*(1), 178–187.

Caves, D. W., Christensen, L. R., & Diewert, W. E. (1982). The economic theory of index numbers and the measurement of input, output, and productivity. *Econometrica, 50*(6), 1393–1414.

Chambers, R. G. (2002). Exact nonradial input, output, and productivity measurement. *Economic Theory, 20*(4), 751–765.

Chambers, R. G., Chung, Y., & Färe, R. (1996). Benefit and distance functions. *Journal of Economic Theory, 70*(2), 407–419.

Chambers, R. G., Chung, Y., & Färe, R. (1998). Profit, directional distance functions, and Nerlovian efficiency. *Journal of Optimization Theory and Applications, 98*(2), 351–364.

Chambers, R. G., Färe, R., & Grosskopf, S. (1996). Productivity growth in APEC countries. *Pacific Economic Review, 1*(3), 181–190.

Chambers, R. G., & Pope, R. D. (1996). Aggregate productivity measures. *American Journal of Agricultural Economics, 78*(5), 1360–1365.

Chavas, J. P., & Cox, T. (1999). A generalized distance function and the analysis of production efficiency. *Southern Economic Journal, 66*(2), 294–318.

Daraio, C., & Simar, L. (2016). Efficiency and benchmarking with directional distances: A data-driven approach. *Journal of the Operational Research Society, 67*(7), 928–944.

Diewert, W. E. (2005). Index number theory using differences rather than ratios. *American Journal of Economics and Sociology, 64*(1), 347–395.

Färe, R., Grosskopf, S., Lindgren, B., & Roos, P. (1995). Productivity developments in swedish hospitals: A Malmquist output index approach. In A. Charnes, W. W. Cooper, A. Y. Lewin, & L. M. Seiford (Eds.), *Data envelopment analysis: Theory, methodology and applications* (pp. 253–272). Kluwer, Boston.

Färe, R., Grosskopf, S., & Lovell, C. A. K. (1994). *Production frontiers*. Cambridge, Cambridge University Press.

Färe, R., Grosskopf, S., & Margaritis, D. (2008). Efficiency and productivity: Malmquist and more. In H. Fried, C. A. K. Lovell, & S. Schmidt (Eds.), *The measurement of productive efficiency and productivity change* (pp. 522–621). New York, Oxford University Press.

Färe, R., & Lovell, C. A. K. (1978). Measuring the technical efficiency of production. *Journal of Economic Theory, 19*(1), 150–162.

Kerstens, K., Mounir, A., & de Woestyne, I. V. (2012). Benchmarking mean-variance portfolios using a shortage function: The choice of direction vector affects rankings!. *Journal of the Operational Research Society, 63*(9), 1199–1212.

Kerstens, K., Shen, Z., & de Woestyne, I. V. (2018). Comparing Luenberger and Luenberger-Hicks-Moorsteen productivity indicators: How well is total factor

productivity approximated? *International Journal of Production Economics, 195*, 311–318.
Kerstens, K., & de Woestyne, I. V. (2014). Comparing Malmquist and Hicks–Moorsteen productivity indices: Exploring the impact of unbalanced vs. balanced panel data. *European Journal of Operational Research, 233*(3), 749–758.
Layer, K., Johnson, A. L., Sickles, R. C., & Ferrier, G. D. (2020). Direction selection in stochastic directional distance functions. *European Journal of Operational Research, 280*(1), 351–364.
Lovell, C. A. K. (2003). The decomposition of Malmquist productivity indexes. *Journal of Productivity Analysis, 20*(3), 437–458.
Luenberger, D. G. (1992a). Benefit functions and duality. *Journal of Mathematical Economics, 21*(5), 461–481.
Luenberger, D. G. (1992b). New optimality principles for economic efficiency and equilibrium. *Journal of Optimization Theory and Applications, 75*(2), 221–264.
Luenberger, D. G. (1995). *Microeconomic Theory*. McGraw Hill, Boston.
McFadden, D. (1978). Cost, revenue, and profit functions. In M. Fuss, & D. McFadden (Eds.), *Production economics: A dual approach to theory and applications, volume 1* (pp. 3–109). Amsterdam: North-Holland Publishing Company.
Mehdiloozad, M., Sahoo, B. K., & Roshdi, I. (2014). A generalized multiplicative directional distance function for efficiency measurement in DEA. *European Journal of Operational Research, 232*(3), 679–688.
Pastor, J. T., & Aparicio, J. (2010). Distance functions and efficiency measurement. *Indian Economic Review, 45*(2), 193–231.
Pastor, J. T., Lovell, C. A. K., & Aparicio, J. (2020). Defining a new graph inefficiency measure for the proportional directional distance function and introducing a new malmquist productivity index. *European Journal of Operational Research, 281*(1), 222–230.
Ray, S. C., & Desli, E. (1997). Productivity growth, technical progress, and efficiency change in industrialized countries: Comment. *American Economic Review, 87*(5), 1033–1039.
Russell, R. R. (1988). On the axiomatic approach to the measurement of technical efficiency. In W. Eichhorn (Ed.), *Measurement in economics: Theory and application of economic indices* (pp. 207–217). Heidelberg: Physica-Verlag.
Russell, R. R. (2018). Theoretical productivity indices. In E. Grifell-Tatjé, C. A. K. Lovell, & R. Sickles (Eds.), *The oxford handbook on productivity analysis* (pp. 153–182). New York, Oxford University Press.
Russell, R. R., & Schworm, W. (2009). Axiomatic foundations of efficiency measurement on data-generated technologies. *Journal of Productivity Analysis, 31*(2), 77–86.
Russell, R. R., & Schworm, W. (2011). Properties of inefficiency indexes on ⟨input, output⟩ space. *Journal of Productivity Analysis, 36*(2), 143–156.
Shephard, R. W. (1970). *Theory of cost and production functions*. Princeton: Princeton University Press.
Zieschang, K. D. (1984). An extended Farrell efficiency measure. *Journal of Economic Theory, 33*(2), 387–396.
Zofío, J. L. (2007). Malmquist productivity index decompositions: A unifying framework. *Applied Economics, 39*(18), 2371–2387.
Zofío, J. L., Pastor, J. T., & Aparicio, J. (2013). The directional profit efficiency measure: On why profit inefficiency is either technical or allocative. *Journal of Productivity Analysis, 40*(3), 257–266.