Asia-Pacific Journal of Operational Research
Vol. 31, No. 1 (2014) 1450010 (13 pages)

DOI: 10.1142/S0217595914500109

SOLUTION METHODS FOR NONCONVEX FREE DISPOSAL HULL MODELS: A REVIEW AND SOME CRITICAL COMMENTS*

KRISTIAAN KERSTENS†
CNRS-LEM (UMR 8179), IESEG School of Management
3 rue de la Digue, F-59000 Lille, France
k.kerstens@ieseg.fr

IGNACE VAN DE WOESTYNE
KU Leuven, Research Unit MEES
Warmoesberg 26, B-1000 Brussels, Belgium
ignace.vandewoestyne@kuleuven.be

Received 6 May 2013
Revised 23 August 2013
Accepted 26 August 2013
Published 10 December 2013

This note first succinctly summarizes the currently available methods to solve the various nonconvex free disposal hull (FDH) models for technical efficiency as well as for minimum costs. It also offers some empirical illustration as to their computational efficiency. Second, this note briefly points out that the recent article by Keshvari and Dehghan Hardoroudi (2008) and its correction by Alirezaee and Khanjani Shiraz (2010) proposing an extended enumeration method to solve for technical efficiency evaluated relative to this family of FDH models contain no original results.

Keywords: Free disposal hull; enumeration method.

1. Introduction

While convexity of technology is traditionally maintained in the non-parametric approach to production theory (Afriat, 1972 or Diewert and Parkan, 1983), a non-convex series of technologies and cost functions have been developed in the literature. This stream in the literature is known under the moniker free disposal hull (FDH) models, a family of nonconvex variations on the more widely used convex data envelopment analysis (DEA) models. A basic nonconvex FDH imposing the assumptions of free (strong) disposal of inputs and outputs has probably first been proposed in Deprins *et al.* (1984) (though Afriat (1972) already mentioned the single output case).

*The most helpful comments of a referee are gratefully acknowledged. The usual disclaimer applies.
†Corresponding author

Two extensions on this basic FDH model occurred in the literature. Firstly, specific returns to scale assumptions have been introduced into this basic model and a new goodness-of-fit method to infer the characterization of returns to scale for nonconvex technologies has been proposed in Kerstens and Vanden Eeckaut (1999). Simplifications in this goodness-of-fit method to characterize returns to scale have been proposed in Soleimani-damaneh *et al.* (2006) and Soleimani-damaneh and Reshadi (2007). Soleimani-damaneh and Mostafaee (2009) offer some stability intervals for preserving the latter classification of returns to scale via a polynomial-time algorithm based on the calculation of certain ratios of inputs and outputs. Second, these nonconvex production models have been complemented by nonconvex cost functions with corresponding specific returns to scale assumptions in Briec *et al.* (2004).

Obviously, apart from the above basic model extensions one can mention a whole series of methodological refinements and variations that have been introduced in the literature. Most often, these proposals are related to methods initially developed in a convex setting. First, Van Puyenbroeck (1998) introduced a nonconvex super-efficiency model, while Sun and Hu (2009) compare in total three methods (including super-efficiency) to discriminate among FDH-efficient observations. Second, Mairesse and Vanden Eeckaut (2002) add lower and upper bound restrictions to scaling on the nonconvex production models.[a] Third, already Tulkens (1993) suggested a free replicability hull (FRH) by allowing for integer replications of all observations (eventually complemented by upper bounds on the integer replication process). This FRH is computationally quite challenging (see Ehrgott and Tind, 2009). Green and Cook (2004) alternatively defined a nonconvex technology containing all observations as well as all composite observations obtained by simple aggregation and call it a free coordination hull (FCH). This FCH can eventually also be complemented by an upper bound on the number of observation being aggregated. Fourth, given the potentially huge amounts of slacks and surpluses associated with traditional radial efficiency measures in FDH models, non-radial efficiency measures have been evaluated and found particularly relevant in the basic FDH model by De Borger *et al.* (1998). Finally, some studies have introduced specific models to cope with various measurement scales. For instance, Jahanshahloo *et al.* (2004) develop proper nonconvex models to handle interval data, while Triantis and Girod (1998) offer a fuzzy mathematical programming approach to measure technical efficiency when production plans are fuzzy (not crisp) for the basic FDH model. There are also some models that are harder to classify: for instance, following Podinovski (2005), Leleu (2009) proposes new formulations combining aspects of convex and nonconvex production models alike for all returns to scale assumptions. Obviously, this is but a selective sample of nonconvex model extensions currently available in the literature.

[a]Bouhnik *et al.* (2001) earlier developed lower bound restrictions on the intensity variables to avoid unreasonable optimal solutions that are compatible with both convex and nonconvex models, though these authors only elaborate on the convex case.

Though it is clear that these nonconvex technology and cost models are nowhere as popular as their convex counterparts, a rather substantial amount of studies have employed the basic FDH model and its extensions. A selective series of examples include: Alam and Sickles (2000) study the dynamics of technical efficiency (TE) following the deregulation of the US airline industry; Walden and Tomberlin (2010) compute convex and nonconvex plant capacity estimates in fisheries; De Witte and Marques (2011) explore scale economies in nonconvex technologies; Cummins and Zi (1998) contrast convex and nonconvex estimates of TE and cost functions; Destefanis and Sena (2005) investigate productivity change using nonconvex technologies etc.

A major point is that the results of nonconvex technology and cost models are often very different from their convex counterparts. It is well-known that nonconvex technology models yield lower inefficiency levels and larger amounts of efficient observations compared to convex models. For instance, Walden and Tomberlin (2010) obtain mean plant capacity estimates that vary between 52% and 84% in the convex and the nonconvex case, respectively. Needless to say that such differences have potentially huge implications when designing policies to combat overcapacity in fisheries. Perhaps more importantly, the seminal article of Briec *et al.* (2004) and the study of Cummins and Zi (1998), for example, report diverging convex and nonconvex cost estimates: from the latter study, one can infer that convex cost estimates are on average a staggering 49.46% below the nonconvex ones (under variable returns to scale).

Given these substantial empirical differences and the fact that the convexity axiom can only be justified in terms of time-divisibility (ignoring any setup times, but also ignoring indivisibilities, increasing returns to scale, positive or negative production externalities, etc. that each can lead to nonconvexity) and is thus chiefly maintained for analytical convenience (Hackman, 2008), one may wonder why these methods are relatively speaking still so little used in economics and related disciplines. Note that in engineering certain production processes are known to be inherently nonconvex and/or nonlinear and requiring appropriate models. A case in point is the so-called economic dispatch problem minimizing total fuel costs of electricity generation subject to various unit and system constraints (some recent examples include Ravi *et al.*, 2006; Park *et al.*, 2010; Tsai *et al.*, 2011).

Why is this evidence from one discipline ignored in other parts of the literature? One main reason may be the desire for theoretical consistency related to the fact that the main duality relations in economics linking production and cost approaches presume convexity (Hackman, 2008). Another reason is simply computational: nonconvex production and cost models are more difficult to solve. The latter is exactly the reason why we think it is time to take stock of the currently available solution methods.

The purpose of this short note is twofold. First, it reviews and clarifies the available solution methods to solve radial measures of TE and cost functions relative to these various FDH models and offers some illustration as to their computational efficiency. Second, it makes a few critical comments by pointing out that some articles in this journal as well as some others do not contain new results. In particular, the articles by Keshvari and Dehghan Hardoroudi (2008) and the ensuing correction by Alirezaee and Khanjani Shiraz (2010) develop an extended enumeration method to compute radial TE measures with respect to FDH models with various returns to scale assumptions. However, these enumeration methods have all been presented in earlier publications.

This note is structured as follows. Section 2 introduces in detail the definitions of the FDH technologies with various returns to scale assumptions and briefly indicates the corresponding cost functions. The next Sec. 3 reviews the existing solution methods for computing efficiency measures and cost functions relative to these nonconvex technologies. Then, we illustrate the time gains when using implicit enumeration relative to the use of linear programming. A final section concludes.

2. FDH Technologies and Cost Functions: Definitions

Denoting an n-dimensional input vector ($x \in \mathbb{R}^n_+$) and an m-dimensional output vector ($y \in \mathbb{R}^m_+$), the set of production possibilities or technology S is defined as follows: $S = \{(x, y) \in \mathbb{R}^{n+m}_+ : x \text{ can produce } y\}$. The input set related to this technology S contains all input vectors x capable to produce a certain output vector y: $L(y) = \{x \in \mathbb{R}^n_+ : (x, y) \in S\}$.

The radial input efficiency measure can be defined as:

$$DF_i(x, y) = \min\{\lambda : \lambda \geq 0, \lambda x \in L(y)\}. \tag{1}$$

Its main properties are: (i) $0 < DF_i(x, y) \leq 1$, with efficient production on the boundary (isoquant) of $L(y)$ represented by unity; (ii) it has a cost interpretation (for instance, Hackman, 2008). Turning to a dual representation of technology, the cost function defines the minimum costs to produce an output vector y given a vector of semi-positive input prices ($w \in \mathbb{R}^n_+$):

$$C(y, w) = \min\{w \cdot x : x \in L(y)\}. \tag{2}$$

A unified algebraic representation of convex and nonconvex technologies under different returns to scale assumptions for a sample of K observations is (Briec *et al.*, 2004):

$$S^{\Lambda,\Gamma} = \left\{(x, y) : x \geq \sum_{k=1}^{K} x_k \delta z_k, y \leq \sum_{k=1}^{K} y_k \delta z_k, z_k \in \Lambda, \delta \in \Gamma\right\}, \tag{3}$$

where

(i) $\Gamma \equiv \Gamma^{\text{CRS}} = \{\delta : \delta \geq 0\}$;
(ii) $\Gamma \equiv \Gamma^{\text{NDRS}} = \{\delta : \delta \geq 1\}$;
(iii) $\Gamma \equiv \Gamma^{\text{NIRS}} = \{\delta : 0 \leq \delta \leq 1\}$;
(iv) $\Gamma \equiv \Gamma^{\text{VRS}} = \{\delta : \delta = 1\}$; and

(i) $\Lambda \equiv \Lambda^{\mathrm{C}} = \{\sum_{k=1}^{K} z_k = 1 \text{ and } z_k \geq 0\}$ and
(ii) $\Lambda \equiv \Lambda^{\mathrm{NC}} = \{\sum_{k=1}^{K} z_k = 1 \text{ and } z_k \in \{0,1\}\}$.

There is one activity vector (z) operating subject to a convexity (C) or nonconvexity (NC) constraint and a scaling parameter (δ) allowing for a particular scaling of all K observations determining the technology. This scaling parameter is free under constant returns to scale (CRS), smaller than or equal to one or larger than or equal to one under non-increasing returns to scale (NIRS) and non-decreasing returns to scale (NDRS), respectively, and fixed at unity under variable returns to scale (VRS). For reasons of space, the corresponding cost functions are not formally defined, but these follow directly from minimizing costs (2) relative to these technologies (3).

The advantages of this formulation are twofold. First, it provides a unified formulation of all basic technologies under different returns to scale assumptions and under both convexity and nonconvexity. Second, it has a pedagogical advantage in that it clearly separates the role of the various assumptions in the formulation of the technology. The restrictions on the scaling parameter (δ) are directly related to the definitions of the axioms on returns to scale in technologies (Hackman, 2008). Equally so, the sum constraint on the activity vector z (i.e., constraint (Λ^{C})) is related to the convexity axiom (Hackman, 2008). This formulation therefore avoids confusing statements that are abound in the DEA literature. For instance, the sum constraint on the activity vector z (i.e., constraint (Λ^{C})) in the envelopment or primal formulation is often called a "convexity constraint" under the VRS assumption, while the CRS technology has no such constraint despite it maintaining the convexity axiom (see, e.g., Cook and Seiford, 2009, pp. 2–3).

3. FDH Technologies and Cost Functions: Solution Methods

Starting our overview of the developments in solution methods to compute these nonconvex FDH production models and their related cost functions, we first start with the computational methods for obtaining the radial input efficiency measure (1) and then discuss the methods to obtain the cost functions (2). Each time, we first treat the traditional convex case followed by the less widely used nonconvex case.

Computing the radial input efficiency measure (1) relative to convex technologies in (3) requires solving a nonlinear programming problem (NLP) for each evaluated observation. However, Briec and Kerstens (2006) show how this NLP can be transposed into the familiar linear programming (LP) problem around in the literature (Hackman, 2008).[b]

[b] By substituting $w_k = \delta z_k$ in (3), one can rewrite the sum constraint on the activity vector. Realizing that the constraints on the scaling factor are in fact integrated into the latter sum constraint, the traditional LP appears (see Briec and Kerstens, 2006 Lemma 2.1 for details).

For the nonconvex technologies in (3), nonlinear mixed integer programs (NLMIP) must be solved in the initial formulation of Kerstens and Vanden Eeckaut (1999). Three distinctive alternative solution methods have followed this initial state of affairs. First, Podinovski (2004) reduces the computational complexity by reformulating all these nonconvex technologies as mixed integer programs (MIP) using a big M technique.[c] Second, starting from an existing LP model for the basic FDH model with VRS (Agrell and Tind, 2001), Leleu (2006) takes this one step further by formulating equivalent LP problems. Third, Briec *et al.* (2004) develop an implicit enumeration strategy for all nonconvex technologies in (3) to obtain closed form solutions. Briec and Kerstens (2006) refine this analysis and furthermore indicate that the computational complexity of this enumeration is advantageous compared to these previous proposals to use MIP or LP. Note that the use of enumeration for the basic nonconvex FDH production model with VRS (i.e., constraints Γ^{VRS} and Λ^{NC}) has been around in the literature since quite a while (examples include Deprins *et al.*, 1984; Lovell, 1995; Tulkens, 1993, among others).[d]

Apart from the fact that the development of both the MIP approach and the enumeration approaches for these nonconvex technologies coincide temporally while the LP approach has come next, it is immediately clear that the Keshvari and Dehghan Hardoroudi (2008) and Alirezaee and Khanjani Shiraz (2010) articles are preceded by both Briec *et al.* (2004) and Briec and Kerstens (2006) articles in terms of the development of implicit enumeration algorithms. While the Keshvari and Dehghan Hardoroudi (2008) article cites the Tulkens (1993) article as an early precursor and fundamentally starts of from the Leleu (2006) article to derive its extended numeration methods, this article ignores the fact that Leleu (2006, p. 341) explicitly cites the existence of the implicit enumeration algorithms in Briec *et al.* (2004).[e] Therefore, it is not surprising that the Keshvari and Dehghan Hardoroudi (2008) article reports results that have been earlier published in Briec *et al.* (2004) and Briec and Kerstens (2006). To be precise, apart from some minor and obvious notational differences, the key results in Keshvari and Dehghan Hardoroudi (2008, Theorem on p. 693) correspond to the results in Briec *et al.* (2004, Proposition 2), apart from the NDRS case (i.e., constraint Γ^{NDRS}) which has been established in Alirezaee and Khanjani Shiraz (2010, p. 608) but which coincides again with Briec *et al.* (2004, Proposition 2).

Turning to the computation of the cost function (2) relative to convex nonparametric technologies, it is well-known that this involves solving one LP per observation being evaluated (Hackman, 2008). Jahanshahloo *et al.* (2007b, 2008) simplify these LP formulations by cutting down on the amount of constraints and

[c] The general difficulty of obtaining a good choice of big M is well-known (Camm *et al.*, 1990).
[d] This is also acknowledged in the Keshvari and Dehghan Hardoroudi (2008) article.
[e] Leleu (2006, p. 341) states: "There are two computational methods used to solve the FDH models. The first one is based on enumeration algorithms as proposed by Tulkens (1993), Cherchye *et al.* (2001) or Briec *et al.* (2004). The second one is the use of mathematical programming."

decision variables. Jahanshahloo *et al.* (2007a) develop cost efficiency (CE) models for the particular case of ordinal data.

For the cost functions (2) relative to the nonconvex technologies in (3), Briec *et al.* (2004) develop implicit enumeration algorithms (Proposition 3). Later on, Leleu (2006) offers an LP solution related to the LP formulations of the relevant technologies discussed supra. Paryab *et al.* (2012) reduce the complexity of these LP formulations of FDH-CE with various returns to scale axioms by reducing both the number of constraints and decision variables.[f] Finally, Mostafaee (2011) develops bounds on nonconvex revenue (and by extension cost) functions for the case of imprecise data without solving any LP.

This terminates our brief history of the developments in computing nonconvex FDH models of production and cost in the OR as well as the economics literature. The relative merits of using an enumeration versus an LP approach in nonconvex models when computing TE have been discussed in Briec and Kerstens (2006, pp. 143–144). While the number of arithmetic operations for enumeration is about $O(LK(n+m)^2)$, with L a measure of data storage for a given precision, an LP has a $O(Lr^3)$ polynomial time complexity for r variables, whereby we assume the use of the most successful interior point method so far available (i.e., primal-dual Newton step interior point method).

We now turn to a simple empirical illustration documenting the computational gains of implicit enumeration compared to LP and MIP approaches.

4. LP and MIP Versus Enumeration Methods: An Illustration

The sample is taken from the Journal of Applied Econometrics Data archive to guarantee duplication purposes.[g] The sample from the Atkinson and Dorfman (2009) article contains monthly data for several years on 16 hydro-electric power plants in Chile. Focusing on the single year 1997, we ignore technical change by specifying an inter-temporal frontier, which results in 192 ($= 12 \times 16$) observations in total. The single output is electricity generated and the three inputs are labor, capital and water. Except for the input capital, all remaining flow variables are expressed in physical units. There are also prices for these three inputs which are denominated in current Chilean pesos. Basic descriptive statistics for the inputs and the single output as well as more details on these data are available in Atkinson and Dorfman (2009).

Computing TE (1) and the cost function (2) relative to the basic VRS FDH model in (3) using implicit enumeration, LP and MIP yields the basic descriptive statistics reported in Table 1. Note that it is well-known that TE relative to a convex

[f] In fact, as pointed out by a referee it is trivial to show that the only contribution of Paryab *et al.* (2012) is already found in Jahanshahloo *et al.* (2007b, Theorem 1: Part (iii)). However, the latter article is regrettably ignored in the former article.
[g] See `http://qed.econ.queensu.ca/jae/`.

Table 1. Nonconvex technical and cost efficiencies.

	TE	CE
Average	0.9555	0.5789
St. deviation	0.1378	0.3565
Minimum	0.3154	0.0548
Maximum	1.0000	1.0000

technology is smaller or equal to the TE evaluated to a corresponding nonconvex technology (i.e., $DF_i^C(x,y) \leq DF_i^{NC}(x,y)$). It is equally obvious that the convex CE estimate is lower or equal to its nonconvex counterpart (i.e., $C^C(y,w)/w \cdot x \leq C^{NC}(y,w)/w \cdot x$).

More important for our purpose, the CPU time in seconds for computing the TE and cost function solutions using LP, MIP and implicit enumeration are reported in Table 2. All computations are performed with the 64-bits version of Maple version 16 on a desktop computer with a 64-bit Intel Core Xeon processor running at 3.60 GHz. Table 2 lists the total time as well as descriptive statistics on the time per observation. The differences in CPU time between the enumeration method and the LP and MIP methods are substantial both at the sample level and for individual observations.

Contrary to what one might expect, the LP method performs worse than the MIP method despite Maple making use of an interior point method. The main reason for this observation comes from the hugely sized LP model of Leleu (2006). To compute TE for one observation, $2K$ decision variables (in casu 384) and $(n+m)K$ inequality constraints yielding slack variables (in casu 768) are needed leading to a constraint matrix of dimensions $((n+m)K+1)\times(n+m+2)K$ (in casu 769×1152) compared to $K+1$ decision variables (in casu 193) and $m+n+1$ general linear constraints (in casu 5) in the MIP model. In the cost function computations, this even increases to $(n+2)K$ decision variables (in casu 960) and $(n+m+1)K$ slack variables (in casu 960) in the LP model resulting in a matrix of dimensions $((n+m+1)K+1)\times(2n+m+3)K$ (in casu 961×1920) contrary to $K+n$ decision variables (in casu 195) and $n+m+1$ general linear constraints (in casu 5) in the

Table 2. CPU time under LP, MIP and enumeration methods.

	LP		MIP		Enumeration	
	TE	$C(y,w)$	TE	$C(y,w)$	TE	$C(y,w)$
Total	78.9560	138.1039	4.1242	3.8838	0.1831	0.6364
Average	0.4112	0.7193	0.0215	0.0202	0.0010	0.0033
St. deviation	0.0247	0.0275	0.0035	0.0026	0.0004	0.0011
Minimum	0.3234	0.6535	0.0172	0.0164	0.0003	0.0026
Maximum	0.4642	0.7855	0.0328	0.0271	0.0049	0.0115

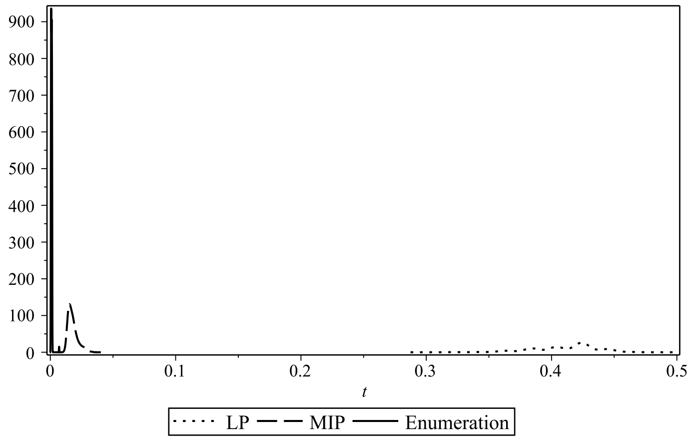

Fig. 1. Kernel density estimates of CPU time for TE.

MIP model.[h] Moreover, the MIP models only make use of binary integer variables making the branch and bound tree rather simple.

To better appreciate the results, Fig. 1 plots kernel density estimates of CPU time for TE using LP, MIP and enumeration: clearly the entire distribution for enumeration is situated way below the distributions for LP and MIP and has much smaller variation. For the cost function estimates, a figure with a similar basic shape results (not displayed).

5. Concluding Comments

This short note has offered a succinct taxonomy of methods available in the economics and OR literatures to solve the various FDH models for TE as well as for minimum costs. It thereby has pointed out that the Keshvari and Dehghan Hardoroudi (2008) article and the ensuing correction by Alirezaee and Khanjani Shiraz (2010) do not contain new results. An empirical illustration underscored the very substantial computational gains of implicit enumeration algorithms compared to MIP and especially to LP.

While it is obvious that implicit enumeration algorithms have a computational advantage, it is good to end with several caveats. First, for certain other purposes, including duality analysis and sensitivity analysis, the use of LP has obvious advantages compared to MIP and implicit enumeration. For instance, imagine one would be computing a credit-constrained profit model (e.g., a nonconvex version of

[h]For comparison, in the convex case the LP model for computing TE only needs $K+1$ decision variables (in casu 193) and $n+m+1$ slack variables (in casu 5) yielding a matrix of dimensions 6×198 while only $K+n$ decision variables (in casu 195) and $n+m+1$ slack variables (in casu 5) are required for computing the cost function leading to matrix of dimension 6×200.

Blancard *et al.* (2006)), then it would be useful not only to know whether the credit-constraint is binding or not, but also to obtain its shadow price when binding. It remains to be seen whether some of these methods can be judiciously combined to speed up the LP approach to answer these type of questions.

Second, for practitioners needing specialized versions of the above basic nonconvex production and cost models (for example, a short-run or sub-vector radial input efficiency measure defined on some of the input dimensions solely) or being interested in a non-standard type of production or economic model (e.g., non-radial efficiency measures, or the nonconvex version of the credit-constrained profit model mentioned supra), these needs are unlikely covered in available software packages. Therefore, it may be easier to program an LP in a standard optimization software rather than derive the implicit enumeration algorithm required. To neutralize this convenience advantage of LP, it would be necessary to come up with a general formulation of these implicit enumeration algorithms covering a wide variety of special production and cost models (e.g., apart from different returns to scale, also different measurement orientations of efficiency, different sub-vector cases, revenue and profit functions in addition to cost functions, etc): such general formulation is currently lacking. This challenge also ignores the methodological refinements and variations listed in Sec. 1.

Third, while there is some literature deriving certain visualizations of convex technologies (for instance, isoquants in input space, transformation curves in output space, etc.) using parametric programming (Krivonozhko *et al.*, 2004), no such literature exists for the nonconvex case. Obviously, it is straightforward to transpose these existing parametric programming approaches to the LP approach to nonconvex production and cost models. It remains an open question whether one can come up with enumeration algorithms facilitating this job.

Finally, our discussion has so far been limited to static production models solely. Recently, Soleimani-damaneh (2013) introduced the first nonconvex dynamic cost function model around in the literature. He manages to rely on a recursive form of implicit enumeration algorithms alone. It remains to be seen whether future developments in this specific research avenue will lead to new computational challenges or not.

In conclusion, while enumeration algorithms may have obvious advantages compared to LP and other optimization methods, it remains to be seen whether enumeration algorithms can be extended beyond the standard computations of efficiency and cost functions in nonconvex production and cost models. This constitutes a major challenge for future research.

References

Afriat, S (1972). Efficiency estimation of production functions. *International Economic Review*, 13(3), 568–598.

Agrell, P and J Tind (2001). A dual approach to nonconvex frontier models. *Journal of Productivity Analysis*, 16(2), 129–147.

Alam, I and R Sickles (2000). Time series analysis of deregulatory dynamics and technical efficiency: The case of the US airline industry. *International Economic Review*, 41(1), 203–218.

Alirezaee, M and R Khanjani Shiraz (2010). A note on an extended numeration method for solving free disposal hull models in DEA. *Asia-Pacific Journal of Operational Research*, 27(5), 607–610.

Atkinson, S and J Dorfman (2009). Feasible estimation of firm-specific allocative inefficiency through Bayesian numerical methods. *Journal of Applied Econometrics*, 24(4), 675–697.

Blancard, S, J-P Boussemart, W Briec and K Kerstens (2006). Short- and long-run credit constraints in French agriculture: A directional distance function framework using expenditure-constrained profit functions. *American Journal of Agricultural Economics*, 88(2), 351–364.

Bouhnik, S, B Golany, U Passy, S Hackman and D Vlatsa (2001). Lower bound restrictions on intensities in data envelopment analysis. *Journal of Productivity Analysis*, 16(3), 241–261.

Briec, W and K Kerstens (2006). Input, output and graph technical efficiency measures on non-convex FDH models with various scaling laws: An integrated approach based upon implicit enumeration algorithms. *TOP*, 14(1), 135–166.

Briec, W, K Kerstens and P Vanden Eeckaut (2004). Non-convex technologies and cost functions: Definitions, duality and nonparametric tests of convexity. *Journal of Economics*, 81(2), 155–192.

Camm, J, AS Raturi and S Tsubakitani (1990). Cutting big M down to size. *Interfaces*, 20(5), 61–66.

Cook, W and L Seiford (2009). Data envelopment analysis (DEA) thirty years on. *European Journal of Operational Research*, 192(1), 1–17.

Cummins, D and H Zi (1998). Comparison of frontier efficiency methods: An application to the U.S. life insurance industry. *Journal of Productivity Analysis*, 10(2), 131–152.

De Borger, B, G Ferrier and K Kerstens (1998). The choice of a technical efficiency measure on the free disposal hull reference technology: A comparison using US banking data. *European Journal of Operational Research*, 105(3), 427–446.

De Witte, K and R Marques (2011). Big and beautiful? On non-parametrically measuring scale economies in non-convex technologies. *Journal of Productivity Analysis*, 35(3), 213–226.

Deprins, D, L Simar and H Tulkens (1984). Measuring labor efficiency in post offices. In *The Performance of Public Enterprises*: *Concepts and Measurements*, M Marchand, P Pestieau and H Tulkens (Eds.), pp. 243–268. Amsterdam: North Holland.

Destefanis, S and V Sena (2005). Public capital and total factor productivity: New evidence from the Italian regions, 197098. *Regional Studies*, 39(5), 603–617.

Diewert, W and C Parkan (1983). Linear programming test of regularity conditions for production functions. In *Quantitative Studies on Production and Prices*, W Eichhorn, K Neumann and R Shephard (Eds.), pp. 131–158. Würzburg: Physica-Verlag.

Ehrgott, M and J Tind (2009). Column generation with free replicability in DEA. *Omega*, 37(5), 943–950.

Green, R and W Cook (2004). A free coordination hull approach to efficiency measurement. *Journal of the Operational Research Society*, 55(10), 1059–1063.

Hackman, S (2008). *Production Economics*: *Integrating the Microeconomic and Engineering Perspectives*. Berlin: Springer.

Jahanshahloo, G, R Kazemi Matin and A Hadi Vencheh (2004). On FDH efficiency analysis with interval data. *Applied Mathematics and Computation*, 159(1), 47–55.

Jahanshahloo, G, M Soleimani-damaneh and A Mostafaee (2007a). Cost efficiency analysis with ordinal data: A theoretical and computational view. *International Journal of Computer Mathematics*, 84(4), 553–562.

Jahanshahloo, G, M Soleimani-damaneh and A Mostafaee (2007b). On the computational complexity of cost efficiency analysis models. *Applied Mathematics and Computation*, 188(1), 638–640.

Jahanshahloo, G, M Soleimani-damaneh and A Mostafaee (2008). A simplified version of the DEA cost efficiency model. *European Journal of Operational Research*, 184(2), 814–815.

Kerstens, K and P Vanden Eeckaut (1999). Estimating returns to scale using nonparametric deterministic technologies: A new method based on goodness-of-fit. *European Journal of Operational Research*, 113(1), 206–214.

Keshvari, A and N Dehghan Hardoroudi (2008). An extended numeration method for solving free disposal hull models in DEA. *Asia-Pacific Journal of Operational Research*, 25(5), 689–696.

Krivonozhko, V, O Utkin, A Volodin, I Sablin and M Patrin (2004). Constructions of economic functions and calculations of marginal rates in DEA using parametric optimization methods. *Journal of the Operational Research Society*, 55(10), 1049–1058.

Leleu, H (2006). A linear programming framework for free disposal hull technologies and cost functions: Primal and dual models. *European Journal of Operational Research*, 168(2), 340–344.

Leleu, H (2009). Mixing DEA and FDH models together. *Journal of the Operational Research Society*, 60(12), 1730–1737.

Lovell, C (1995). Measuring the macroeconomic performance of the Taiwanese economy. *International Journal of Production Economics*, 39(1), 165–178.

Mairesse, F and P Vanden Eeckaut (2002). Museum assessment and FDH technology: Towards a global approach. *Journal of Cultural Economics*, 26(4), 261–286.

Mostafaee, A (2011). Non-convex technologies and economic efficiency measures with imprecise data. *International Journal of Industrial Mathematics*, 3(4), 259–275.

Park, J-B, Y-W Jeong, J-R Shin and KY Lee (2010). An improved particle swarm optimization for nonconvex economic dispatch problems. *IEEE Transactions on Power Systems*, 25(1), 156–166.

Paryab, K, R Khanjani Shiraz and L Jalalzadeh (2012). An improved model for assessing the FDH-cost efficiency. *Asia-Pacific Journal of Operational Research*, 29(4), 1250022.

Podinovski, V (2004). On the linearisation of reference technologies for testing returns to scale in FDH models. *European Journal of Operational Research*, 152(3), 800–802.

Podinovski, V (2005). Selective convexity in DEA models. *European Journal of Operational Research*, 161(2), 552–563.

Ravi, G, R Chakrabarti and S Choudhuri (2006). Nonconvex economic dispatch with heuristic load patterns using improved fast evolutionary program. *Electric Power Components and Systems*, 34(1), 37–45.

Soleimani-damaneh, M (2013). An enumerative algorithm for solving nonconvex dynamic DEA models. *Optimization Letters*, 7(1), 101–115.

Soleimani-damaneh, M, G Jahanshahloo and M Reshadi (2006). On the estimation of returns-to-scale in FDH models. *European Journal of Operational Research*, 174(2), 1055–1059.

Soleimani-damaneh, M and A Mostafaee (2009). Stability of the classification of returns to scale in FDH models. *European Journal of Operational Research*, 196(3), 1223–1228.

Soleimani-damaneh, M and M Reshadi (2007). A polynomial-time algorithm to estimate returns to scale in FDH models. *Computers & Operations Research*, 34(7), 2168–2176.

Sun, S and S-C Hu (2009). Discriminating efficient units using MAJ FDH. *Applied Mathematics and Computation*, 215(8), 3116–3123.

Triantis, K and O Girod (1998). A mathematical programming approach for measuring technical efficiency in a fuzzy environment. *Journal of Productivity Analysis*, 10(1), 85–102.

Tsai, M-T, H-J Gow and W-M Lin (2011). A novel stochastic search method for the solution of economic dispatch problems with non-convex fuel cost functions. *Electrical Power and Energy Systems*, 33(4), 1070–1076.

Tulkens, H (1993). On FDH efficiency analysis: Some methodological issues and applications to retail banking, courts, and urban transit. *Journal of Productivity Analysis*, 4(1–2), 183–210.

Van Puyenbroeck, T (1998). Some remarks on modified FDH. *Journal of Productivity Analysis*, 9(1), 81–94.

Walden, J and D Tomberlin (2010). Estimating fishing vessel capacity: A comparison of nonparametric frontier approaches. *Marine Resource Economics*, 25(1), 23–36.

Kristiaan Kerstens received a PhD in Economics (1994) from Katholieke Universiteit Brussel (nowadays KU Leuven), Belgium. Since 1999 he works as research professor at CNRS in Lille, France (LEM, UMR 8179). He is also professor of economics at IESEG School of Management in Lille, France. His work covers productivity indices and indicators, Data Envelopment Analysis (DEA) and related models, and multi-moment portfolio optimization. His more than 50 articles have been published in such outlets as: Bulletin of Economic Research, Economic Modeling, Economic Theory, European Journal of Operational Research, International Journal of Production Economics, Journal of Banking & Finance, Journal of Economic Dynamics and Control, Journal of the Operational Research Society, Journal of Optimization Theory and Applications, Journal of Productivity Analysis, Management Science, Omega, Transportation Research, among others.

Ignace Van de Woestyne received a PhD in Mathematics (1989) from KU Leuven, Belgium. Since 1989, he works as professor (full professor since 2005) at the faculty of Economics and Business at Hogeschool-Universiteit Brussel, recently integrated in KU Leuven, Belgium. After doing research in differential geometry and computer vision earlier, his research interests have shifted toward mathematical modeling in production economics (non-parametric methods) and in finance (multi-moment portfolio optimization). His articles have been published in Applied Economics, Economic Modeling, European Journal of Operational Research, Journal of Banking & Finance, Journal of the Operational Research Society, Kuwait Journal of Science and Engineering, Tôhoku Journal of Mathematics, among others. He is also co-editor of several volumes of the series "Geometry and Topology of Submanifolds" published by World Scientific Publishing.