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Benchmarking mean-variance portfolios using a shortage function: the choice of direction vector affects rankings!

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In addition to its use in data envelopment analysis models, the shortage function has been proposed as a tool to gauge performance in multi-moment portfolio models. An open issue is how the choice of direction vector affects the efficiency measurement, especially when some of the data can be negative and, from a practical point of view, whether and how the resulting league tables are affected. This paper illustrates empirically how the choice of direction vector affects the relative ranking of mean-variance portfolios. This result is relevant to all frontier-based applications, especially those where some of the data can be naturally negative.

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1. Introduction

Traditionally, the performance of portfolios within the mean-variance framework has been evaluated using performance measures which include some combination of information on both return and risk. Well-known classic examples are the Sharpe, Treynor and Jensen measures, among others (see Grinblatt and Titman (1989) for a critical discussion). Obviously, the menu of available portfolio performance measures has meanwhile been substantially expanded (see, eg, Feibel (2003) or Bacon (2008) for surveys).

In production theory, the shortage function has been introduced by Luenberger (1995) and generalises existing distance functions. In particular, it simultaneously looks for reductions in inputs and expansions in outputs and is dual to the profit function.[1] Inspired by these developments in micro-economic theory where distance functions are used to characterise choice sets and to establish duality relations with value functions, Briec *et al* (2004) introduce the shortage function as a portfolio performance gauging tool in the traditional mean-variance (MV) portfolio framework. In particular, these authors show that this shortage function can represent the MV-space and serves as an efficiency measure to position any portfolio with respect to the boundary of this portfolio set. Furthermore, developing a dual framework to assess the degree of satisfaction of investors preferences, they propose a decomposition of portfolio performance into allocative and portfolio efficiency. This approach is extended to the mean-variance-skewness (MVS) space in Briec *et al* (2007) where the shortage function seeks to project portfolios onto the MVS-frontier by looking for potential improvements (ie, increasing return and skew, while decreasing risk). A framework for the general moment portfolio problem is described in Briec and Kerstens (2010), where the shortage function seeks to magnify odd moments and to decrease even moments.

This new shortage function approach has meanwhile been applied to hedge funds in a variety of contributions (see, eg, Bacmann and Benedetti, 2009, or Jurczenko and Yanou, 2010). It has also been contrasted to a series of alternative portfolio models in Lozano and Guttiérez (2008), among others. This shortage function has been compared to alternative distance functions to lay the foundations for geometric reconstructions of MVS efficient portfolio sets in Kerstens *et al* (2011a). Finally, it has been employed to define a discrete time Luenberger indicator that can separate between performance changes due to portfolio strategies on the one hand, and performance changes due to the market evolution on the other hand (see Brandouy *et al*, 2010).

Apart from portfolio theory, the utilisation of non-parametric frontier estimators in conjunction with the

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[1]Similar functions define duality relations in consumption theory (see Luenberger, 1995).

use of distance functions (that also have an efficiency interpretation) has found its way in a variety of finance topics. The performance of mutual funds has been rated along a multitude of dimensions (rather than mean and variance solely) using these boundary estimators. For instance, the seminal article of Murthi *et al* (1997) employs return as a desirable output to be increased and risk and a series of transaction costs as an input to be reduced, and measures the performance of each mutual fund with respect to a piecewise linear frontier (rather than a traditional non-linear portfolio frontier). Most recently, Kerstens *et al* (2011b) have argued for using the shortage function to critically assess mutual funds and these authors have systematically tested for the need of higher order moments in defining these efficiency measures. In the context of asset selection and following the innovative article of Alam and Sickles (1998), similar ideas have been employed to show that changes in productive efficiency at least partially translate into changes in stock performance (see Edirisinghe and Zhang (2008) and Nguyen and Swanson (2009) for recent contributions).

All works (from Briec *et al* (2004) to Briec and Kerstens (2010)) project any portfolio onto the portfolio frontier by using a direction vector directly related to the observed position of the portfolio itself. This choice of direction vector results in a proportional efficiency measure, which is convenient for practitioners. Notice that less general distance functions have been proposed in an MV portfolio context (eg, Morey and Morey (1999) propose a risk contraction measure and a return expansion measure). Therefore, it is natural to ask how this choice of direction for this general distance function affects the efficiency assessment of portfolios. Intuitively, most researchers would assume that an efficiency measure has the same interpretation independent of the position of the observation being evaluated. This would then supposedly lead to curves indicating an identical level of (in)efficiency that—admittedly vaguely stated—run somewhat parallel to the boundary of the frontier. However, in some experiments with the shortage function in a two-dimensional MV-space, we were struck by the fact that this is not necessarily guaranteed.

This observed problem might be related to the fact that returns and other odd moments in portfolio theory can be negative, while production is normally confined to semi-positive input and output vectors. While in the non-parametric efficiency literature in production some articles do treat the case of negative numbers occurring in some particular specifications (eg, growth in employment, losses instead of profits, etc), this problem is omnipresent in a portfolio context. The work by, for example, Silva Portela *et al* (2004) specifically deals with negative data when using an adapted kind of shortage function (see Pastor and Ruiz (2007) for a summary of these issues in the data envelopment analysis (DEA) literature and Portela and Thanassoulis (2010) for a recent application in a productivity measurement context). In recent work, Kerstens and Van de Woestyne (2011) show that the shortage function can be straightforwardly generalised to handle negative data while maintaining a proportional interpretation under weak conditions.

Thus, while the topic we want to document is illustrated using the generalised shortage function in a traditional two-dimensional MV-space, it might also be of great relevance to the whole DEA literature dealing with negative data.[2] However, in this DEA literature most specifications contain several variables and this multidimensional nature makes it much harder to illustrate the above stated phenomenon.[3] Thus, as extensively empirically documented in this paper, in MV portfolio analysis, the generalised shortage function (Kerstens and Van de Woestyne, 2011) seems to lead to problems of performance measurement around the origin, a problem that hitherto went unnoticed in the literature. In particular, the impact of the different choices of direction vector is far from innocent, since apart from the efficiency measures also the resulting ranks of portfolios are substantially affected. To the best of our knowledge, this phenomenon related to the use of efficiency measures in a context where some data can be negative has never been documented in any frontier-based publication.

Evident starting points to look for eventual readily available answers is the literature on the axiomatic foundations of efficiency measures in production theory and some further related literature in operations research. The analysis of the axiomatic foundations of efficiency measures in production theory goes back to at least Färe and Lovell (1978). Färe and Lovell (1978) initially proposed three axioms that an input-based efficiency index should satisfy: (i) indication (the index equals unity if and only if the input vector belongs to the strongly efficient subset), (ii) monotonicity (for constant other inputs and outputs, increasing an input must reduce the value of the index), and (iii) homogeneity of degree minus one (doubling inputs must halve the index). Later on, additional axioms have been proposed (eg, Russell (1985, 1990) suggested (iv) invariance for units of measurement (commensurability), and (v) continuity in technology and also in input or output quantities). A recent article summarising this axiomatic literature is Russell and Schworm (2009).

This literature can prove only partially inspiring, since it focuses mainly on special distance functions that merely

[2] In principle, the same issues could be illustrated using geometric reconstructions of MVS efficient portfolio sets (see Kerstens *et al*, 2011a), but this would come at a substantial computational cost and the visual interpretation would be less straightforward.

[3] Although recently quite some progress has been made in visualising properties of DEA frontiers: see, eg, Førsund *et al* (2009).

look for reduction in inputs (or improvements in outputs). However, the shortage function measures potential efficiency improvements in all dimensions. Russell and Schworm (2011) recently took a look at similar efficiency measures in production theory and prudently conclude that the shortage function with a proportional interpretation satisfies a stronger unit invariance property compared to the case of a fixed direction. Furthermore, as may be clear from the above description so far, the traditional distance functions analysed in this axiomatic literature do not normally adhere to the mathematical notion of a norm. An exception is, for example, the general article by Briec and Leleu (2003) treating the use of a Hölder distance function in a DEA production context. There is some other DEA literature adopting mathematical norms to resolve issues of infeasibility related to so-called super-efficiency models among others (see, eg, Jahanshahloo *et al*, 2004 or Amirteimoori *et al*, 2005) or the stability of efficiency classifications (see, eg, Takeda and Nishino, 2001).[4] But, none of these articles using mathematical norms in DEA models seems to have documented the above stated phenomenon. In brief, many alternatives to the proportional shortage function are conceivable, but this existing work remains silent on how to choose among the existing options and did never document the impact of these choices on efficiency-based rankings.

Therefore, in this contribution we intend to systematically explore the consequences of choosing different direction vectors for the generalised shortage function in an MV-portfolio context. In particular, this contribution is organised as follows. Section 2 provides some basic definitions as well as extensions in line with the framework developed in Briec *et al* (2004). Section 3 offers a structured study of different choices of the direction vector in a basic MV context. In Section 4, making use of visualisations, the effect of these different choices on the rankings of portfolios is documented using an empirical example. The final Section 5 concludes and outlines potential future research avenues.

2. Mean-variance portfolio framework

We start by briefly describing the non-parametric MV portfolio framework following the initial Briec *et al* (2004) article, but extending and generalising it where appropriate.

Consider the basic problem of composing a portfolio from an investor's universe consisting of n financial products, the latter referred to as the *financial universe*. A portfolio can then be represented by a weight vector $x=(x_1,\dots,x_n)$. A sum constraint ($\sum_{i=1}^{n} x_i=1$) determines the proportion of each of the initial products. Short selling is excluded, meaning that all weights x_i are assumed to be positive. Consequently, the set of all portfolios, also known as the *portfolio simplex*,[5] is the subset of $\mathbb{R}^n$ determined by

$$\mathfrak{I} = \left\{ x \in \mathbb{R}^n_+; \quad \sum_{i=1}^{n} x_i = 1 \right\}.$$

The expected return vector and covariance matrix of the financial universe can be computed. More precisely, if r_{il} denotes the historical return of the ith financial product in the financial universe ($i=1,\dots,n$) at time l ($l=1,\dots,m$), then the expected return of the ith product over the given time window is equal to

$$R_i = \frac{1}{m}\sum_{l=1}^{m} r_{il}. \tag{1}$$

Furthermore, the covariance between the ith and the jth product, denoted by V_{ij}, is obtained by

$$V_{ij} = \frac{1}{m}\sum_{l=1}^{m} (r_{il}-R_i)(r_{jl}-R_j). \tag{2}$$

The expected return $\mathrm{E}[R(x)]$ of a portfolio x and its variance $\mathrm{Var}[R(x)]$ can be calculated as follows:

$$\mathrm{E}[R(x)] = \sum_{i=1}^{n} x_i R_i, \quad \mathrm{Var}[R(x)] = \sum_{i,j=1}^{n} x_i x_j V_{ij}. \tag{3}$$

Every element of the two-dimensional MV-space $\mathbb{R}^2$ is called an MV-point. Hereafter, an arbitrary MV-point is denoted by its coordinates (v_M, v_V). That is, v_M and v_V are the return and variance components of the MV-point respectively. The portfolio simplex $\mathfrak{I}$ is mapped into the two-dimensional MV-space by means of the function $\Phi:\mathfrak{I}\rightarrow\mathbb{R}^2 : x \mapsto \Phi(x) = (E[R(x)],\ \mathrm{Var}[R(x)])$. The image set $\Phi(\mathfrak{I})$ can be extended to the *disposal representation set* $\mathcal{DR} = \Phi(\mathfrak{I}) + (\mathbb{R}_- \times \mathbb{R}_+)$. This set can be rewritten as follows:

$$\begin{aligned}\mathcal{DR} = \{(v_M,\ v_V) \in \mathbb{R}^2 | \exists x \in \mathfrak{I} : (v_M,\ -v_V) \\ \leqslant (\mathrm{E}[R(x)],\ -\mathrm{Var}[R(x)])\}.\end{aligned} \tag{4}$$

Put differently, $\mathcal{DR}$ consists of all MV-points that are weakly dominated by the image $\Phi(x)$ of some portfolio x. In this perspective, a particular MV-point can thus represent the return and variance of a particular portfolio in the sense that it can be the image by Φ of this portfolio, or can refer to a particular position in this two-dimensional MV-space not related to a real portfolio. We also point out that both $\Phi(\mathfrak{I})$ and $\mathcal{DR}$

[4]In finance, norms are used in a variety of contexts (see, eg, the definition of coherent risk measures in Jarrow and Purnanandam, 2005), but—to the best of our knowledge—never to appraise portfolio performance.

[5]This set of admissible portfolios can be modified to include additional constraints that can be written as linear functions of asset weights (eg, transaction costs): see Briec *et al* (2004). Briec and Kerstens (2010) also consider the cases of a risk-free asset and shorting.

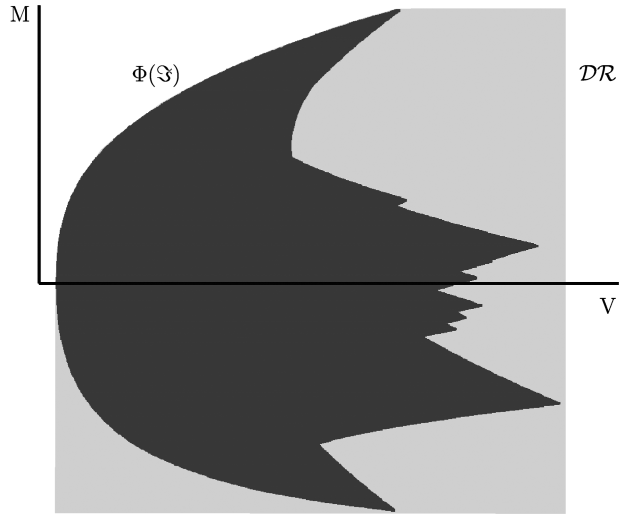


Figure 1 Illustration of the image set $\Phi(\Im)$ and the disposal representation set $\mathcal{DR}$.

are contained in $\mathbb{R} \times \mathbb{R}_+$ since the variance is always positive. As a consequence, both sets are situated in the first and fourth quadrant of the traditional MV-representation. Figure 1 provides an illustration generated from a fictitious financial universe.

To introduce the notion of portfolio efficiency, the *weakly efficient frontier* is defined as a subset of $\mathcal{DR}$ by:

Definition 1 In MV-space, the *weakly efficient frontier* is defined as:

$$\partial^W(\mathcal{DR}) = \{(v_M,\ v_V) \in \mathcal{DR}; (-v'_M,\ v'_V) < (-v_M,\ v_V) \Rightarrow (v'_M,\ v'_V) \notin \mathcal{DR}\}.$$

Clearly, the weakly efficient frontier, also called the *theoretical frontier*, contains all MV-points that are not weakly dominated in MV-space. Notice that by definition, this frontier is part of $\mathcal{DR}$. As the latter is an extension of $\Phi(\Im)$, the theoretical frontier can also contain points not attainable by real portfolios.

The *strongly efficient frontier* is introduced as follows:

Definition 2 In MV-space, the *strongly efficient frontier* is defined as:

$$\partial^S(\mathcal{DR}) = \{(v_M,\ v_V) \in \mathcal{DR}; (-v'_M,\ v'_V) \leqslant (-v_M,\ v_V) \text{ and } (-v'_M,\ v'_V) \neq (-v_M,\ v_V) \Rightarrow (v'_M,\ v'_V) \notin \mathcal{DR}\}.$$

The strongly efficient frontier, shortened to *efficient frontier*, contains all MV-points that are not strictly dominated in MV-space. Now, the extended shortage function is introduced in the following definition:

Definition 3 In MV-space, let $g = (g_M, g_V) \in \mathbb{R}_+ \times \mathbb{R}_-$ and $g \neq 0$. The *extended shortage function* $\mathcal{S}_g$ in the direction of vector g is the function $\mathcal{S}_g : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \cup \{-\infty\}$, with

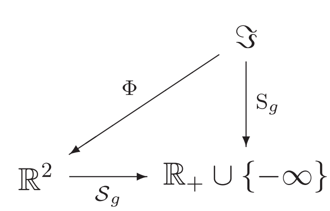
$$\mathcal{S}_g(v) = \sup_{\delta \in \mathbb{R}_+} \{\delta;\ v + \delta g \in \mathcal{DR}\}.$$

The optimal value of δ, if finite, is denoted by δ^*. The corresponding vector $v^* = v + \delta^* g$ is called the *theoretical projected point*.

The direction vector g must look for improvements in return and reduction in risk. As mentioned in the introduction, to guarantee a proportional interpretation most articles in a production context so far take a direction vector g related to the point being projected. But, in a portfolio context and limiting ourselves to a traditional MV-space, returns can be negative. Therefore, following Kerstens and Van de Woestyne (2011) the generalised proportional shortage function capable of handling negative data implies taking the direction vector $g = (|v_M|, -v_V)$ in MV-space. When returns are initially negative, this guarantees one is looking for improvements in the direction of positive returns.

Within this general framework, this contribution explores the impact of opting for a fixed direction vector on the one hand and a position-dependent direction vector on the other hand. The former option moves away from proportionality. Further details on the selected projection schemes are discussed below.

Some further remarks are at stake. Firstly, the value $-\infty$ is only realised if the set $\{\delta;\ v + \delta g \in \mathcal{DR}\}$ is empty since it is commonly accepted that $\sup \varnothing = -\infty$. Secondly, the theoretical projected point v^* is located at the weakly efficient frontier $\partial^W(\mathcal{DR})$. Lastly, the extended shortage function (hereafter merely addressed as shortage function) extends the one introduced by Briec *et al* (2004) and denoted by S_g. The following diagram shows the relation between both functions:



For a given direction vector $g = (g_M, g_V)$ specified as in Definition 3, the shortage function value for an arbitrary MV-point $v = (v_M, v_V)$ under evaluation (an observed portfolio, or some fictitious point) can be computed by solving the following quadratic non-linear model:

$$\begin{aligned} \max_{x,\delta} \quad & \delta \\ & \sum_{i=1}^{n} x_i = 1, \\ \text{s.t.} \quad & \mathrm{E}[R(x)] \geqslant v_M + \delta g_M, \\ & \mathrm{Var}[R(x)] \leqslant v_V + \delta g_V, \\ & \delta \geqslant 0,\ 0 \leqslant x_i \leqslant 1 \text{ for } i \in \{1, \ldots, n\}. \end{aligned} \tag{P1}$$

Indeed, it follows from Definition 3 that the shortage function seeks for the largest possible value δ such that $v + \delta g \in \mathcal{DR}$. Rewriting this condition in terms of coordinates leads to $(v_M + \delta g_M, v_V + \delta g_V) \in \mathcal{DR}$. It then follows from (4) that there must exist a portfolio $x \in \mathfrak{I}$ for which $\Phi(x) = (\mathrm{E}[R(x)], \mathrm{Var}[R(x)])$ weakly dominates $(v_M + \delta g_M, v_V + \delta g_V)$. From this observation, the constraints in Model (P1) follow in a straightforward manner.

Firstly, note that the shortage function can be applied to an arbitrary MV-point. Consequently, an arbitrary MV-point can be the unit of analysis. In particular, this unit of analysis could be the image by Φ of one of the initial financial products in the financial universe generating $\mathcal{DR}$, or might even be the image by Φ of an arbitrary portfolio. Secondly, it follows from Model (P1) that the return level can be considered as a kind of output variable in a production model, whereas the variance level is considered as similar to an input variable. Indeed the goal is to simultaneously maximise return (output) and minimise variance (input).

As for the shortage function value, the following proposition is valid.

Proposition 1 *Let* $v = (v_M, v_V) \in \mathcal{DR}$, *with* $v \neq 0$, *and* g *the direction vector specified in Definition 3. Then* $\mathcal{S}_g(v) \geqslant 0$, *with* $\mathcal{S}_g(v) = 0 \Leftrightarrow v \in \partial^W(\mathcal{DR})$.

Proof Because the set $\{\delta;\ v + \delta g \in \mathcal{DR}\}$ is non-empty ($\delta = 0$ is a member), the result follows directly from Definition 3 of the shortage function. □

From Proposition 1, we conclude that in practical shortage function value computations, three different cases can occur: (i) $\mathcal{S}_g(v) = -\infty$ if $v \notin \mathcal{DR}$, which is observed as an infeasibility when solving model (P1); (ii) $\mathcal{S}_g(v) = 0$ if v is efficient; (iii) $\mathcal{S}_g(v) > 0$ in all other cases, with higher values for less efficient vectors v. This observation makes the shortage function, for instance, suitable for ranking different financial products with respect to some financial universe: a higher shortage function value results in a lower ranking.[6]

[6]To obtain a positive relation between efficiency and the shortage function, one could consider taking the negative of the current definition. However, we prefer to stay in line with current practice in the literature.

Proposition 2 *Let* $g \in \mathbb{R}_+ \times \mathbb{R}_-$ *be a non-zero direction vector. Then,* $\mathcal{S}_{\lambda g} = (1/\lambda)\mathcal{S}_g$ *for an arbitrary* $\lambda > 0$.

Proof For an arbitrary vector v in MV-space, it follows from Definition 3 that

$$\mathcal{S}_{\lambda g}(v) = \sup_{\delta \in \mathbb{R}_+} \{\delta;\ v + \delta\lambda g \in \mathcal{DR}\}.$$

Now, let $\delta' = \delta\lambda$. Then

$$\begin{aligned} \mathcal{S}_{\lambda g}(v) &= \sup_{\delta' \in \mathbb{R}_+} \left\{\frac{\delta'}{\lambda};\ v + \delta' g \in \mathcal{DR}\right\} \\ &= \frac{1}{\lambda} \times \sup_{\delta' \in \mathbb{R}_+} \{\delta';\ v + \delta' g \in \mathcal{DR}\} \\ &= \frac{1}{\lambda}\mathcal{S}_g(v). \quad \square \end{aligned}$$

Proposition 2 demonstrates the effect of scaling the direction vector on the shortage function value. A value $\lambda > 1$ corresponds with an expansion of the direction vector, while $\lambda < 1$ results in a contraction of this vector. Moreover, this proposition also shows that the computation of the shortage function value does not need to be redone after rescaling the direction vector. Indeed, the resulting value can be found directly by multiplying the initial shortage function values with the inverse of the scaling factor.

3. Different choices for the direction vector

The shortage function provides a tool for measuring efficiency of a given portfolio with respect to the weakly efficient MV-frontier of some financial universe. Measuring is done in the direction of the vector g with the purpose to simultaneously increase return and reduce risk. This explains the condition $g = (g_M, g_V) \in \mathbb{R}_+ \times \mathbb{R}_-$ imposed in Definition 3. However, this restriction still leaves ample room for different choices. The literature so far has not described how to determine this direction vector and the impact of this choice on resulting computations (for instance, on efficiency-based rankings of different portfolios).

Inspired by developments in economic theory, several choices can be proposed.[7] One main distinction can be made between (i) a fixed direction, or (ii) a direction determined by the position of the portfolio under evaluation. Additionally, the direction vector could be rescaled or normed. In this section, we discuss each of these options in closer detail.

[7]Briec (1997) introduces a position-dependent direction vector in the efficiency literature. The article by, for example, Chambers *et al* (1998) opts for a fixed direction vector. Also Blackorby and Donaldson (1980) choose a fixed direction vector with unit coordinates when developing absolute inequality measures.

3.1. Fixed direction vector

A fixed direction vector in the shortage function definition is probably the most natural choice to start with. In this subsection, we investigate exactly this case.

Definition 4 The *fixed direction vector*, abbreviated as *FD-vector*, is a vector $g=(c_M, -c_V)\neq(0,0)$ with constants $c_M, c_V \geqslant 0$. Computations of the shortage function value (called *FD-inefficiency*) and related vectors are done according to the *FD-projection scheme* if the FD-vector is used as direction vector.

By definition, the FD-vector points in an increasing return and decreasing risk direction if both c_M and c_V are distinct from zero. Figure 2 illustrates this type of FD-vector in the MV-plane.

Evidently, special directions can be considered (see, eg, Morey and Morey (1999) for similar proposals). If $c_M=0$, then the FD-vector points in a horizontal direction in MV-space. This choice corresponds with a purely risk reducing strategy. The FD-vector points in a vertical upward direction in MV-space if $c_V=0$, corresponding with a purely return increasing strategy.

Apart from the horizontal and vertical directions, plenty of other directions can still be chosen in MV-space. Commonly, one describes intermediate directions by means of an angle. For instance, an equal increase of return and decrease of risk strategy followed in Figure 2 leads to a 135° angle.

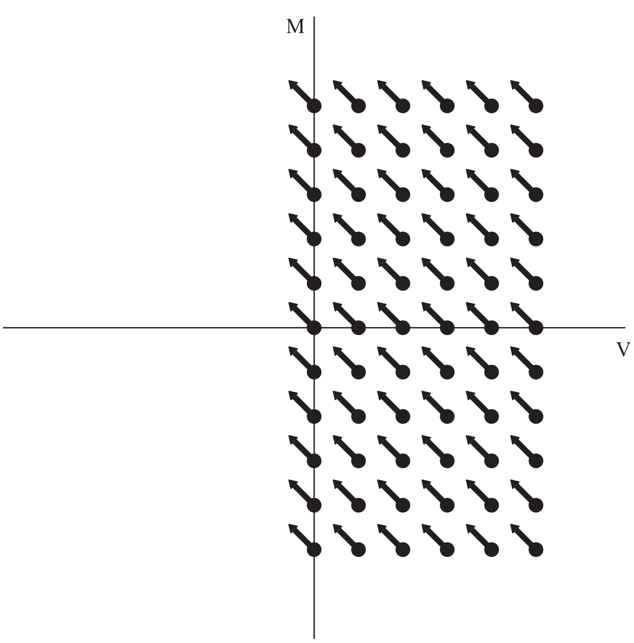


Figure 2 Illustration of the FD-vector pointing to a risk reducing and return increasing direction.

Note that a correct interpretation of a given direction requires the same measuring unit on both axes of the MV-figure. However, most frontier figures in MV-space, of which Figure 1 is an example, are scaled images, thereby optimising printing space and facilitating interpretation. More precisely, usually a different scaling is used on both axes leading to different units, which evidently influences the angle of projection. Consequently, proposing a specific angle of direction should always be accompanied by information on the measuring units to allow for a correct interpretation.

As for the FD-inefficiency, nothing can be added to the general Propositions 1 and 2. These propositions, however, guarantee that the shortage function can be used for ranking financial products on the basis of their efficiency with respect to the MV-frontier. Moreover, the ranking itself is not influenced by rescaling the FD-vector. Only the exact direction of the direction vector needs to be determined. One way to go is to base the choice of direction on investor preferences (in particular, the choice of angle could reflect risk-aversion).[8]

3.2. Unit length fixed direction vector

The FD-vector g in MV-space introduced in the previous subsection is taken rather arbitrarily. In particular, one could consider a vector with unit Euclidean length.[9] This leads to the following definition:

Definition 5 The *unit length fixed direction vector*, abbreviated to *UFD-vector*, is a vector

$$g=\frac{1}{\sqrt{c_M^2+c_V^2}}(c_M, -c_V)\neq(0,0)$$

with constants $c_M, c_V \geqslant 0$. Computations of the shortage function value (called *UFD-inefficiency*) and related vectors are done according to the *UFD-projection scheme* if the UFD-vector is used as direction vector.

Clearly, the UFD-projection scheme is merely a special case of the FD-projection scheme, but with an adjusted direction vector. In fact, the following proposition clarifies the relationship between both projection schemes.

Proposition 3 *Let $g_{FD}=(c_M, -c_V)$ be a FD-direction vector. Then, the UFD-inefficiency equals the FD-inefficiency multiplied by $\sqrt{c_M^2+c_V^2}$.*

[8]Briec *et al* (2004) demonstrate that, due to dual relations between shortage function and mean-variance utility function, the shadow prices associated with the shortage function can yield information about investors risk aversion.

[9]Recall that the Euclidean length of a vector $v=(v_M, v_V)$ is given by $\sqrt{v_M^2+v_V^2}$. Note that instead of Euclidean length, other choices of norms could equally well be considered.

Proof The result follows directly from Proposition 2 and the fact that the corresponding UFD-vector $g_{UFD} = 1\Big/\left(\sqrt{c_M^2 + c_V^2}\right) g_{FD}$. $\square$

Notice that Figure 2 can also serve to illustrate the UPD-projection scheme.[10]

3.3. Position-dependent direction vector

Briec *et al* (2004) opt for a direction vector g depending on the position of the point to be mapped by the shortage function. Their choice, adapted for use with negative data following Kerstens and Van de Woestyne (2011), is explained in the following definition.

Definition 6 Let $v = (v_M, v_V) \in \mathbb{R} \times \mathbb{R}_+$ be a point in MV-space. Then, the *position-dependent direction vector*, abbreviated to *PD-vector*, is the vector $g = (|v_M|, -v_V)$. Computations of the shortage function value (called *PD-inefficiency*) and related vectors are done according to the *PD-projection scheme* if the appropriate PD-vector is used as direction vector.

Clearly, the PD-vector points in a direction simultaneously increasing return and reducing risk. Moreover, this choice provides the shortage function with a convenient proportional interpretation, as can be seen from Proposition 4.[11]

Proposition 4 *Let* $v = (v_M, v_V) \in \mathcal{DR}$ *, with* $v \neq 0$*, and* g *the PD-vector. Then* $0 \leqslant \mathcal{S}_g(v) \leqslant 1$*, with* $\mathcal{S}_g(v) = 0 \Leftrightarrow v \in \partial^W(\mathcal{DR})$.

Proof Because of Proposition 1, we only need to prove additionally that $\mathcal{S}_g(v) \leqslant 1$ for $v = (v_M, v_V) \in \mathcal{DR}$. Clearly, since $v_V \geqslant 0$, $g = (|v_M|, -v_V)$. Consequently,

$$\mathcal{S}_g(v) = \sup_{\delta \in \mathbb{R}_+} \{\delta;\ (v_M, v_V) + \delta(|v_M|, -v_V) \in \mathcal{DR}\}.$$

It now follows that $v_V(1-\delta) \geqslant 0$ which yields the required result. $\square$

Figure 3 illustrates the PD-vector positioned at the corresponding point but without considering the proper length of the vector in an effort not to obscure the image. As can be observed, the PD-vector points towards the origin for original positions located in the fourth quadrant

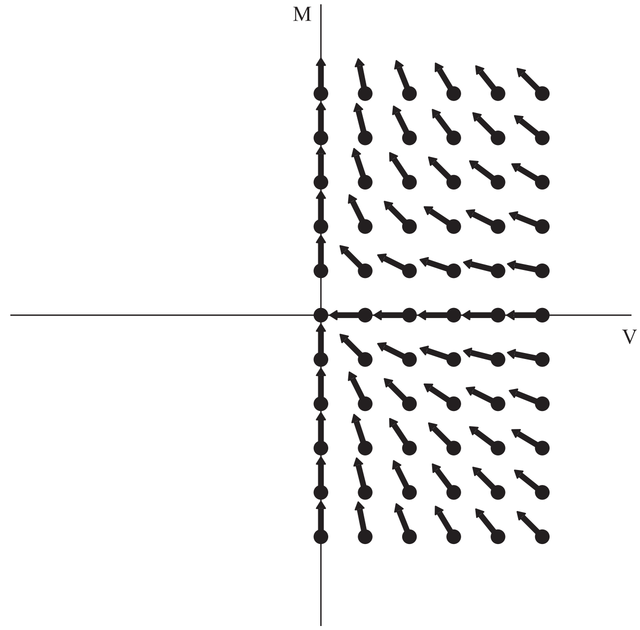


Figure 3 Illustration of the PD-vector, only considering the direction and not the length of the vector, and positioned at the corresponding point.

and points away from the origin for positions situated in the second quadrant. This phenomenon follows directly from the construction. Indeed, if $v = (v_M, v_V)$ is located in the fourth quadrant, then $v_V \geqslant 0$ and $v_M \leqslant 0$. Consequently, $g = (|v_M|, -v_V) = (-v_M, -v_V) = -v$. Similarly, if v is located in the second quadrant, then $v_V \leqslant 0$ and $v_M \geqslant 0$, resulting in $g = (|v_M|, -v_V) = (v_M, v_V) = v$. For points situated in the first or third quadrant, the resulting PD-vector bends away from the origin.

3.4. Unit length position-dependent direction vector

The PD-vector g introduced in the previous subsection is completely determined by the position of the point v in MV-space. In particular, the Euclidean length of the PD-vector varies when the position of v changes. This could be undesirable. Therefore, in this subsection we examine the effect of switching to a direction vector that is still dependent on the position of v but that has unit Euclidean length. We start by introducing the following definition:

Definition 7 Let $v = (v_M, v_V) \in \mathbb{R} \times \mathbb{R}_+$ be a point in MV-space distinct from the origin. Then, the *unit length position-dependent direction vector*, abbreviated to *UPD-vector*, is the vector

$$g = \frac{1}{\sqrt{v_M^2 + v_V^2}} (|v_M|, -v_V).$$

Computations of the shortage function value (called *UPD-inefficiency*) and related vectors are done according

[10] Normally, the arrows in the case of a fixed direction vector and a unit length fixed direction vector would just differ in length. This would hardly be noticeable on a separate figure. To save space, we therefore refer to the same figure.

[11] As a matter of fact, as long as a portfolio model contains an even moment (variance, kurtosis, ..., ie, all observed values in this dimension being strictly positive), this proportional interpretation can be maintained.

to the *UPD-projection scheme* if the appropriate UPD-vector is used as direction vector.

As is the case of the PD-vector, the UPD-vector points in a direction that increases return and decreases risk. In fact, Figure 3 can also perfectly serve as illustration of the UPD-projection scheme. However, the proportionality property (see Proposition 4) is no longer valid when choosing the UPD-vector. Instead, the following holds:

Proposition 5 *Let* $v \in \mathcal{DR}$, *with* $v \neq 0$, *and* g *the UPD-vector. Then,* $0 \leqslant \mathcal{S}_g(v) \leqslant \sqrt{v_M^2 + v_V^2}$. $\mathcal{S}_g(v) = 0 \Leftrightarrow v \in \partial^W(\mathcal{DR})$.

Proof The result follows directly from Propositions 1 and 4. □

4. Empirical application

Several projection schemes have been discussed in Section 3 by means of their properties. In order to understand the differences more thoroughly, we apply these on a set of portfolios obtained from a real database. Visualisations help in understanding the underlying patterns. Moreover, we explain the possible impact of different projection schemes on the efficiency-based ranking of the portfolios under observation for a basic MV portfolio model without shorting and a risk-free rate.

4.1. Data description

The database for this empirical part is obtained from Euronext and contains initially a selection of 101 assets (given by their daily returns from 18 May to 1 October 2009) traded at Euronext Paris. The assets are spread over Large Cap, Middle Cap, Small Cap, Free Market and Bonds. To obtain more detailed visualisations afterwards, this selection has been narrowed down to those assets with an expected return smaller than or equal to 0.006 and a variance less than or equal to 0.001 computed over the given time period. The financial universe on which we proceed thus contains 78 assets.[12]

Contrary to a production setting where the number of decision-making units under evaluation is limited, a financial universe contains infinitely many potentially interesting portfolios. Indeed, as explained in Section 2, every weight vector summing up to one represents a portfolio. Therefore, 20 portfolios labelled from 1 to 20 are randomly drawn from the financial universe provided. Because of the financial universe's size, the portfolio weights are not reported here.

[12]Both the initial database and the proposed selection are available upon simple request from the authors.

4.2. Efficiency computations and discussion

We start by visualising in Figure 4 the MV-frontier and the projection of the proposed portfolios onto this frontier according to different direction schemes. More precisely, we propose the PD- and UPD-projection schemes, and the FD- and UFD-projection schemes. The latter are computed in a horizontal, vertical and slant direction. As slant direction, we propose the FD-vector $g = (M_{\max}, -V_{\max})$, with $M_{\max}$ and $V_{\max}$ the maximal expected return and variance of all assets in the financial universe computed over the given time period, respectively. This choice of direction roughly corresponds, at least for this database, with a projection in a 135° direction when scaled to a square figure (see Figure 4b).

The weakly efficient (black solid curve) and strongly efficient (white dashed curve) MV-frontier, and the left-side boundary of $\Phi(\mathfrak{I})$ (black dotted curve) are visible on the left-hand side of each image of Figure 4. The original assets are visible as grey circles. The selected portfolios (black solid circles) are projected onto their optimal positions (black circles) on the MV-frontier according to the PD- and UPD-projection schemes in Figure 4a, and the FD- and UFD-projection schemes in Figure 4b (slant direction), 4c (horizontal direction), and 4d (vertical direction). Notice that PD and UPD schemes lead to the same projected portfolios. The same remark applies to FD and UFD schemes. Details of these images can be found in Figure 5. Notice that the scaling applied for the latter images visibly results in a different direction in Figure 5b compared with Figure 4b. This serves to illustrate what was mentioned earlier: providing a projection direction in terms of an angle should be accompanied by scaling information for correct interpretation.

Next, the inefficiencies of the 20 portfolios under observation are computed according to the proposed projection schemes. The resulting inefficiencies are reported in Table 1.

Observe that the inefficiencies computed according to FD- and UFD-projection schemes are equal up to a fixed factor. For example, the FD-inefficiency in the slant direction equals the UFD-inefficiency multiplied by 237.82 for all portfolios. Similar factors can be found for the other FD-directions. This result is a mere consequence of Proposition 3.

The noticeable differences in magnitude of different types of inefficiencies make these mutually incomparable. It follows from Proposition 4 that the PD-inefficiency can be interpreted as a proportional inefficiency measure. Focussing on portfolio 10 for instance, the observed PD-inefficiency of 0.883821 can be interpreted as an inefficiency of 88.38%, knowing that all portfolios of the financial universe, and not only those under observation, have a PD-inefficiency between 0% (fully efficient) and 100% (fully inefficient). One might expect a similar

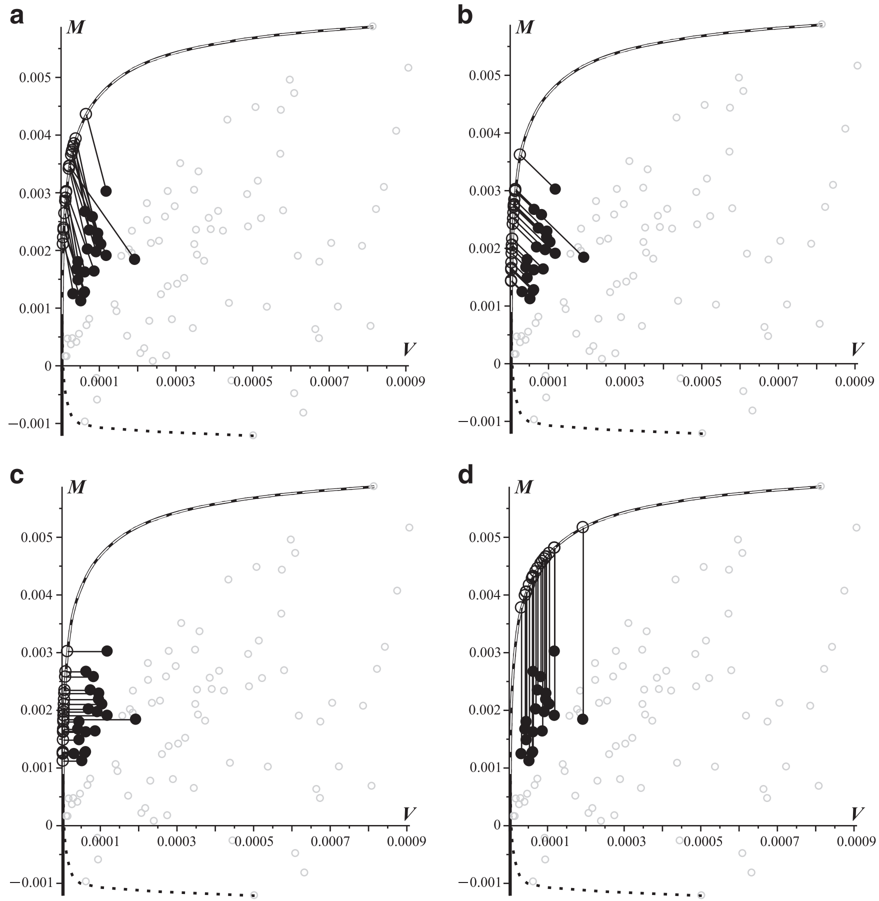


Figure 4 Projection of selected portfolios onto the MV-frontier following different projection schemes. Visualisation of the weakly efficient (black solid curve) and strongly efficient (white dashed curve) MV-frontier, and the left-side boundary of $\Phi(\Im)$ (black dotted curve). The original assets are visible as grey circles. The selected portfolios (black solid circles) are projected onto their optimal positions (black circles) on the MV-frontier according to the following projection schemes: (a) The PD- and UPD-projection scheme; (b) The FD- and UFD-projection scheme in a slant direction; (c) The FD- and UFD-projection scheme in a horizontal direction; (d) The FD- and UFD-projection scheme in a vertical direction.

interpretation for the FD-inefficiency whose magnitude appears to be comparable with that of the PD-inefficiency. Proposition 1, however, is the best one can say, meaning there is no predetermined upper bound for the FD-inefficiencies. Consequently, the FD-inefficiencies cannot be considered as proportional measures.

The UPD- and UFD-inefficiencies are all pretty small. It follows from Definition 3 that, with these projection schemes, the shortage function value can be interpreted as the Euclidean distance to the MV-frontier in the appropriate direction. In fact, this interpretation makes it possible to compare the UPD- and UFD-inefficiencies after all. Taking portfolio 10 again as an example, the slant UFD-inefficiency is equal to 0.000322, while the horizontal and the vertical UFD-inefficiencies equal 0.000050 and 0.003050, respectively. Because of the shape of the MV-frontier (increasing and concave), one can expect that the slant UFD-inefficiency is situated between the horizontal and the vertical UFD-inefficiency. Also the UPD-inefficiency of 0.000990 is situated between the horizontal and vertical UFD-inefficiency. Clearly, this result is valid for all portfolios.

Based on the different inefficiencies computed in Table 1, a ranking of all portfolios is established. The resulting rankings from best (smallest inefficiency) to worst (largest inefficiency) are listed in Table 2.

One is immediately struck by the differences between some of the rankings. Compare, for instance, the PD-ranking with the UPD-ranking. Portfolio 17, for example, performs best in the PD-ranking while its performance is rather average in the UPD-ranking. The differences can even be more pronounced. Portfolio 10, for instance, is the worst possible portfolio in the PD-ranking, which contrasts to its second best position in the UPD-ranking. Other examples show that this behaviour is not in any sense exceptional.

These casual observations are confirmed by Spearman's rank correlations represented in Table 3. Indeed, the rank

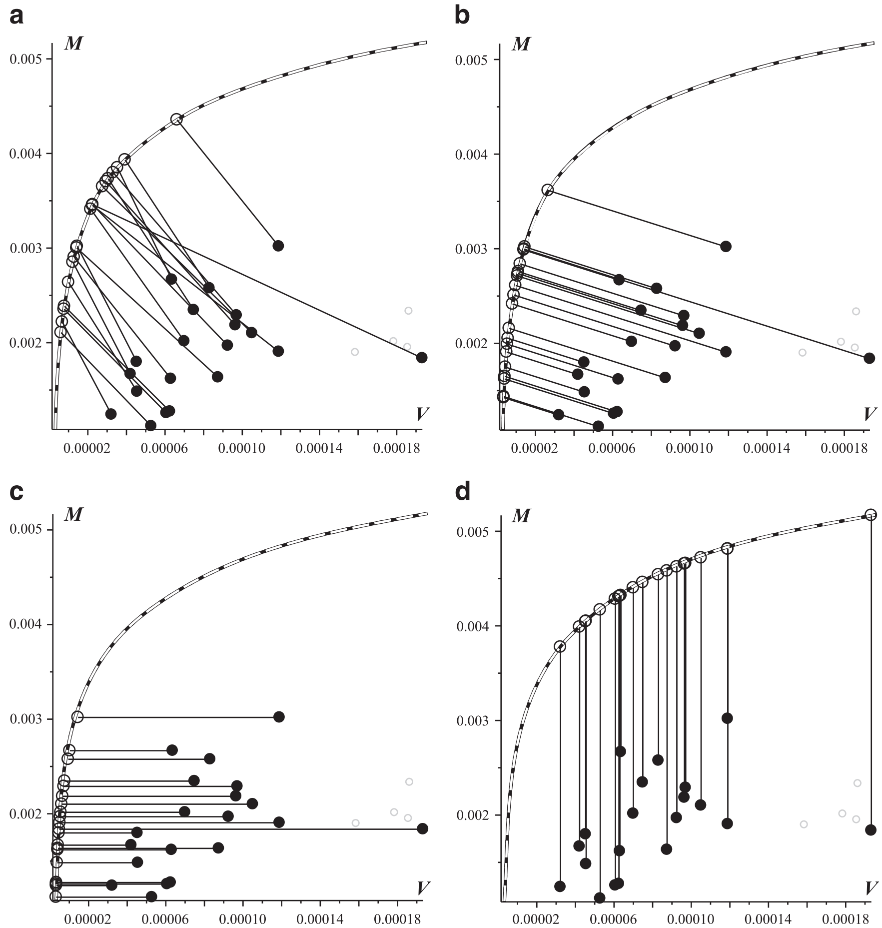


Figure 5 Details of the projection of selected portfolios onto the MV-frontier following different projection schemes. Visualisation of the weakly efficient (black solid curve) and strongly efficient (white dashed curve) MV-frontier. Some of the original assets are visible as grey circles. The selected portfolios (black solid circles) are projected onto their optimal positions (black circles) on the MV-frontier according to the following projection schemes: (a) The PD- and UPD-projection scheme; (b) The FD- and UFD-projection scheme in a slant direction; (c) The FD- and UFD-projection scheme in a horizontal direction; (d) The FD- and UFD-projection scheme in a vertical direction.

correlation between the PD- and UPD-inefficiencies is equal to 0.048 with a high significance of 0.840 meaning that the rank correlation is not statistically significant. Therefore, both rankings can be considered statistically independent. Comparing other rankings, we come to the same conclusion except for the horizontal and the slant FD-UFD rankings. Here, we observe a statistically significant rank correlation of 0.986. Inspection of Table 2 confirms this result since one can notice that only the consecutive portfolios 14 and 17 are interchanged. Consequently, in this example, projecting horizontally or in the slant angle proposed earlier does not result in major differences.

That projecting in different directions can lead to different rankings need not come as a surprise. Indeed, some portfolios can be situated close to the MV-frontier in one direction, but less close in other directions. The real surprise, however, is the drastic effect of switching from a PD-vector to the UPD-vector in the position-dependent projection schemes, since in these cases, the directions are identical for both schemes.

To understand this unexpected behaviour, we visualise the different types of inefficiencies in Figure 6. Starting from the financial universe, a particular rectangular region in MV-space covering $\Phi(\mathfrak{I})$ is identified. On this region, a regular grid of size 200×200 is imposed. For each of the grid points (40 000 in total) the inefficiency measure is computed, following one of the projection schemes. These values are then scaled to a grey tone value that can be used to colour the corresponding grid point. Consequently, the grey tone value gives an indication of the inefficiency value.[13] However, since the human mind is not well trained

[13]This figure exploits the fact that the shortage function is continuous in the portfolio weight vector (x) whenever the direction vector does not contain any zero component (see Briec and Kerstens, 2010: Proposition 2.4).

Table 1 Inefficiencies of 20 randomly selected portfolios according to different projection schemes

Nr	*PD*	*UPD*	*FD hor*	*UFD hor*	*FD ver*	*UFD ver*	*FD slant*	*UFD slant*
1	0.692152	0.001396	0.070869	0.000064	0.406139	0.002387	0.096315	0.000405
2	0.779754	0.001156	0.046033	0.000042	0.436797	0.002567	0.063938	0.000269
3	0.443501	0.001183	0.058670	0.000053	0.281548	0.001655	0.077089	0.000324
4	0.708772	0.001183	0.041753	0.000038	0.394698	0.002320	0.057667	0.000242
5	0.591681	0.001388	0.074082	0.000067	0.359669	0.002114	0.099206	0.000417
6	0.811610	0.001549	0.125345	0.000114	0.494799	0.002908	0.170012	0.000715
7	0.679392	0.001221	0.044759	0.000041	0.382507	0.002248	0.061511	0.000259
8	0.794476	0.001287	0.064905	0.000059	0.458695	0.002696	0.089582	0.000377
9	0.694604	0.001519	0.098996	0.000090	0.420354	0.002470	0.133121	0.000560
10	0.883821	0.000990	0.054667	0.000050	0.519052	0.003050	0.076607	0.000322
11	0.836747	0.001369	0.091765	0.000083	0.501430	0.002947	0.126227	0.000531
12	0.526507	0.001357	0.081001	0.000074	0.334255	0.001964	0.106652	0.000448
13	0.756302	0.001491	0.095947	0.000087	0.451748	0.002655	0.130262	0.000548
14	0.736745	0.001550	0.109030	0.000099	0.445507	0.002618	0.146967	0.000618
15	0.789060	0.000979	0.031841	0.000029	0.431482	0.002536	0.044565	0.000187
16	0.883182	0.001631	0.207546	0.000188	0.566834	0.003331	0.278942	0.001173
17	0.443159	0.001339	0.114992	0.000104	0.305361	0.001795	0.143802	0.000605
18	0.873161	0.001113	0.065143	0.000059	0.516351	0.003034	0.090873	0.000382
19	0.658802	0.001509	0.098974	0.000090	0.403417	0.002371	0.132366	0.000557
20	0.873004	0.001098	0.063138	0.000057	0.514912	0.003026	0.088129	0.000371

Table 2 Ranking of 20 randomly selected portfolios according to different types of inefficiency

Ranked by	*PD*	*UPD*	*FD-UFD hor*	*FD-UFD ver*	*FD-UFD slant*
Best	17	15	15	3	15
	3	10	4	17	4
	12	20	7	12	7
	5	18	2	5	2
	19	2	10	7	10
	7	4	3	4	3
	1	3	20	19	20
	9	7	8	1	8
	4	8	18	9	18
	14	17	1	15	1
	13	12	5	2	5
	2	11	12	14	12
	15	5	11	13	11
	8	1	13	8	13
	6	13	19	6	19
	11	19	9	11	9
	20	9	14	20	17
	18	6	17	18	14
	16	14	6	10	6
Worst	10	16	16	16	16

to discriminate between neighbouring grey tones, an additional transformation is used to cycle 10 times through all grey tone values in consecutive increasing and decreasing order. Put differently, instead of using a pattern consisting of only one grey tone cycle (from black to white), a pattern consisting of 10 cycles (from black to white and back to black) is utilised as colour function. As a consequence, different grey tone bands occur, all determined by MV-points with equal inefficiency, thus, visualising the curves of equal inefficiency. We mention that also the grid points are scaled to obtain square images. To speed-up computations, all inefficiencies have been computed by means of sequential quadratic programming routines identical to the subroutine NPSOL described in Gill *et al* (1986). For the final visualisation, OpenGL rendering techniques have been used. A multi-purpose computer algebra system such as Maple can equally well do the job (however, at the cost of extra processing time).

Comparing Figures 6a and 6b, it becomes clear why switching to a unit length projection vector can have a major impact on the efficiency ranking. Indeed, the lines of equal UPD-inefficiency seem to bend away around the origin, contrary to the lines of equal PD-inefficiency. On the one hand, portfolios with an expected return of nearly zero and higher variance are promoted in the UPD-ranking compared with the PD-ranking. On the other hand, notice that the lines of equal PD-inefficiency in Figure 6a are more condensed near the low variance side of the MV-frontier and close to the origin. This means that the PD-inefficiency changes in this region more drastically than, for instance, in the high variance, high return area. Consequently, it is possible for portfolios to be situated rather close to the MV-frontier and still be considered as rather PD-inefficient. This effect should not be underestimated, especially in the presence of products with very low variance in the financial universe (which is quite often the case). As a consequence, portfolios in that area are almost always very inefficient, which is undesirable. The UPD-projection

Table 3 Spearman's rank correlation between different inefficiency rankings

	PD	*UPD*	*FD-UFD hor*	*FD-UFD ver*	*FD-UFD slant*
PD rank correlation	1.000	0.048	−0.063	0.023	−0.054
Sig. (2-tailed)		0.840	0.791	0.925	0.821
UPD rank correlation	0.048	1.000	−0.314	−0.026	−0.301
Sig. (2-tailed)	0.840		0.177	0.915	0.198
FD-UFD hor rank correlation	−0.063	−0.314	1.000	0.284	0.986**
Sig. (2-tailed)	0.791	0.177		0.225	0.000
FD-UFD ver rank correlation	0.023	−0.026	0.284	1.000	0.293
Sig. (2-tailed)	0.925	0.915	0.225		0.210
FD-UFD slant rank correlation	−0.054	−0.301	0.986**	0.293	1.000
Sig. (2-tailed)	0.821	0.198	0.000	0.210	

**Correlation is significant at the 0.01 level (2-tailed).

scheme tries to correct this effect, but actually overcompensates it as can be seen in Figure 6b.

When analysing Figures 6c, 6d and 6e, one notices that the lines of equal inefficiency are translated in a horizontal, vertical, and slant direction, respectively. The resulting images correspond more closely to the idea that portfolios closer to the MV-frontier should have a higher efficiency ranking. Especially the slant FD- (and UFD-) projection scheme live up to this expectation. Remark that the horizontal FD- (and UFD-) projection concentrates the lines of equal inefficiency near the high variance, high return area of the MV-frontier (see Figure 6c). A similar behaviour is noticeable in the low variance, low return area of the MV-frontier when projecting vertically according to a FD- (and UFD-) projection scheme (see Figure 6d).

5. Conclusions and recommendations

In Section 3, we have described possible projection schemes for computing efficiency by means of the generalised shortage function. From the empirical analysis in Section 4, we have learned that different projection schemes can lead to different portfolio rankings. The PD-projection scheme has an advantage that its inefficiency has a proportional interpretation (ie, having a value between zero and one), thereby making it suitable for situating the efficiency of any portfolio with respect to the financial universe. However, as a major drawback we observe that, in the presence of low variance products in the financial universe, portfolios with a low variance and low return have a rather high PD-inefficiency. Therefore, these types of portfolios are conceived as being inefficient. Put differently, the PD-inefficiency of high variance, high return products is less sensitive to changes compared to low variance, low return products. Consequently, high variance, high return products are usually promoted in PD-rankings.

The UPD-projection scheme tries to correct the drawback in the PD-projection scheme but actually overcompensates by bending the lines of equal UPD-inefficiency away near the origin of the MV-plane. As a consequence, portfolios with a higher variance but a return close to zero are over-promoted in the UPD-ranking. This effect can equally lead to undesirable effects in the ranking. The UPD-inefficiency has the advantage, however, that it can be interpreted as a true Euclidean distance to the MV-frontier in the appropriate direction. Therefore, its interpretation concurs with the natural idea of inefficiency as some kind of distance to the frontier: smaller distances to the frontier result in smaller inefficiency values.

The FD- and UFD-projection schemes lead to identical rankings, at least for the same fixed direction. When applying a UFD-projection scheme, the corresponding UFD-inefficiency can be interpreted as a Euclidean distance to the MV-frontier measured in that direction. However, the FD- and UFD-inefficiencies cannot be interpreted as proportional inefficiency measures. The computation of one inefficiency value can, therefore, not be interpreted, which should not be a problem when applied for ranking portfolios on the basis of their inefficiency.

To summarise, all projection schemes considered here seem to have their specific advantages and drawbacks in an MV portfolio context. Therefore, an ideal projection scheme is not apparent at this moment. When it comes to efficiency-based ranking of portfolios, however, we are in favour of the slant UFD-projection scheme since (a) the UFD-inefficiency can be interpreted as a Euclidean distance to the MV-frontier, which is therefore interpretable (ie, points closer to the frontier are more efficient); (b) the lines of equal UFD-inefficiency in Figure 6e correspond most to our expectations (ie, these basically have a similar shape as the original frontier).

This paper is, to the best of our knowledge, the first in systematically examining the effect of different projection

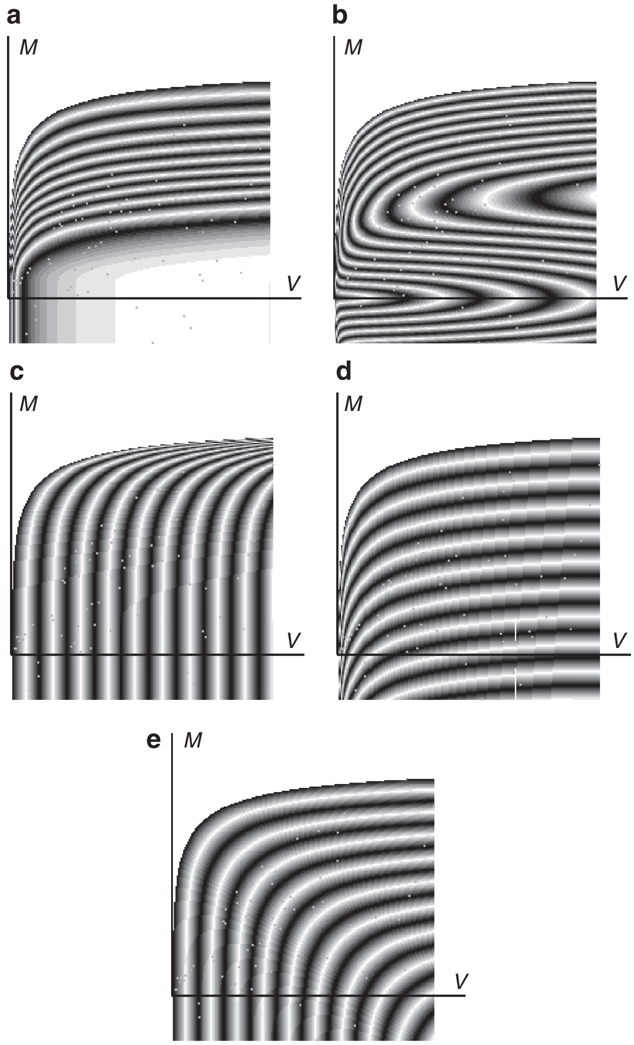


Figure 6 Visualisation of the curves of equal inefficiency according to different projection schemes. For each point of a 2d-grid covering $\Phi(\Im)$, the inefficiency measure is computed, following different projection schemes. The inefficiency values obtained are scaled cyclical to a grey tone value for colouring the corresponding grid point. In this way, the curves of equal inefficiency become visible. The following projection schemes are applied: (a) The PD-projection scheme; (b) The UPD-projection scheme; (c) The FD- and UFD-projection scheme in a horizontal direction; (d) The FD- and UFD-projection scheme in a vertical direction; (e) The FD- and UFD-projection scheme in a slant direction. The original assets are visible as small grey dots.

schemes for the generalised shortage function in an MV portfolio setting. Clearly, this research is not finished. It could, for instance, be interesting to look for the mathematical reasons why the lines of equal UPD-inefficiency bend away near the origin of the MV-space. Furthermore, instead of using the Euclidean distance, other distance norms can be considered. The effect of other norm choices on the corresponding efficiencies and on related rankings is unknown so far. Finally, it could be equally interesting to examine the results of this analysis in multi-dimensional piece-wise linear DEA production models where some data can be naturally negative (eg, growth rates). At the theoretical level, it may be useful to explore the use of distance functions adhering to several mathematical norms and their corresponding dual value functions in both production and portfolio contexts.[14]

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[14]A bit similar to Briec and Lesourd (1999) who prove that some Hölder normed distance function is dual to the profit function.

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