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ORIGINAL RESEARCH

Nonradial plant capacity concepts: proposals and attainability

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Abstract

This contribution observes that plant capacity notions based on traditional radial efficiency measures may leave substantial amounts of slacks or unmeasured inefficiency. These unmeasured inefficiencies can result in inaccurate assessments of production capabilities, potentially leading to suboptimal operational and strategic decisions. To remedy this problem, we define new nonradial output-oriented and input-oriented plant capacity concepts based on nonradial Färe-Lovell efficiency measures. By leveraging nonradial measures, our approach captures multidimensional inefficiencies, providing a more nuanced and accurate evaluation of production performance across various input and output dimensions. Furthermore, we also explore how the introduction of nonradial attainability levels can render the attainable output-oriented plant capacity concept more flexible. This flexibility allows for the incorporation of realistic operational constraints, ensuring that capacity assessments are both practical and adaptable to diverse production environments. An empirical illustration on a secondary data set illustrates the pertinent differences between radial and nonradial plant capacity notions. Our empirical analysis demonstrates that nonradial measures offer a more detailed understanding of capacity utilization. In particular, it shows that nonradial plant capacity concepts are especially important on a nonconvex technology.

Keywords Capacity utilization · Slacks · Färe-Lovell efficiency measure · Convexity · Nonconvexity · Attainability

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1 Introduction

The concept of plant capacity, initially introduced in the production literature by Johansen (1968), has since been rigorously examined and operationalized by Färe et al. (1989) and Fare et al. (1989). The former focused on a single output scenario while the latter delved into scenarios with multiple outputs within a non-parametric frontier framework. This conceptualization has found extensive application in various empirical contexts, including but not limited to fisheries (e.g., Walden and Tomberlin (2010)), hospitals (e.g., Karagiannis (2015)), and banking sectors (e.g., Sahoo and Tone (2009)). Such diverse applications underscore the versatility and relevance of the plant capacity concept across different domains.

Over the past two decades, plant capacity utilization (PCU) measures have predominantly relied on a pair of output-oriented (O-O) radial efficiency metrics, as summarized by Fukuyama et al. (2021). Tingley et al. (2003) propose an O-O radial PCU measure under non-increasing return to scale. More recently, Cesaroni et al. (2017) introduce an input-oriented (I-O) PCU measure, defined as the ratio of two I-O radial efficiency measures, following a similar approach. Empirical studies comparing O-O and I-O PCU notions have been conducted by Chen and Kerstens (2023), Kerstens and Shen (2021), and Shen et al. (2022), among others. Cesaroni et al. (2019) propose novel long-run O-O PCU measures and theoretically and empirically compared short- and long-run PCU measures. The latter PCU notions have been empirically applied by, for example, Kerstens and Shen (2021), Shen et al. (2022), and Song et al. (2023).

Kerstens et al. (2019a) conducted a comparative analysis of O-O and I-O technical and economic PCU measures, investigating the influence of convexity on these metrics. Their empirical findings indicate that convexity plays a significant role in nearly all technical and economic PCU measures. Within conventional parametric, semi-parametric, and nonparametric technology frameworks, the incorporation of convexity is a common practice. The prevalent adherence to the convexity axiom in mainstream literature often stems from its convenience or from the assumption of perfect time divisibility. In-depth discussions regarding convexity and nonconvexity in both nonparametric technologies and cost functions are extensively covered in Kerstens and Van de Woestyne (2021). The profound impact of convexity on technologies is apparent, as highlighted by empirical evidence presented in Kerstens and Van de Woestyne (2021). Moreover, beyond mere cost differentials, convexity also influences economies of scale, as elucidated by their empirical analysis.

A notable repercussion of convexity on technology, extensively documented in literature, is the tendency of traditional radial efficiency measures to leave considerable slack and surplus variables. However, under nonconvexity, these measures can result in even greater amounts of slack and surplus variables. This observation has been corroborated by studies conducted on the same dataset of USA banks, with Ferrier et al. (1994) documenting this phenomenon in the convex case, and De Borger et al. (1998) observing it in the nonconvex case.

Hence, there arises a potential challenge associated with PCU measures employing radial efficiency measures. Radial efficiency measures assess performance relative to isoquants rather than to efficient subsets, potentially misidentifying some technically inefficient units as efficient (Färe & Lovell, 1978). Within the context of capacity utilization, this characteristic can erroneously label plant capacity as optimal when, in reality, it is not. Generally, the optimal capacity determined by O-O radial efficiency measures may be underestimated, while that determined by I-O radial efficiency measures may be overestimated. Consequently, the primary objective of this study is to introduce novel nonradial PCU measures, incorporating both input and output orientations. These measures are designed based on the premise that optimal capacity should be situated within the efficient subset rather than merely on the isoquant.

It is pertinent to mention that several nonradial PCU measures have been introduced in the literature utilizing nonradial efficiency measures in the literature. Segerson and Squires (1990) define a partial PCU measure in a parametric framework. This measure has been transposed by Vestergaard et al. (2003) in a nonparametric context using an asymmetric Färe efficiency measure. These authors focus on radial expansion of one specific output while keeping other outputs unchanged. Recently, a generalized capacity measure based on a couple of directional distance functions is developed by Yang and Fukuyama (2018). Kerstens et al. (2020) introduce a novel graph PCU measure based on the generalized Farrell graph efficiency measure. However, these nonradial measures still ignore the effects of slacks on plant capacity measurement, i.e., the inference of optimal capacity in plant capacity measurement may be incorrect because of the existence of slacks. The proposed new nonradial plant capacity measures fill this void by using weighted Färe-Lovell efficiency measures that fully consider all slacks simultaneously.

The attainability of variable inputs is an endogenous issue from the definition of PCU suggested by Johansen (1968). To explore this issue, an attainable O-O PCU measure is defined by Kerstens et al. (2019b). They also highlight that the attainability issue only exists for the O-O PCU measure, while the I-O PCU measure does not suffer from it. A recent empirical application of the attainable O-O PCU notion is found in Cui et al. (2023).

Hence, this attainability issue is explored in the new nonradial O-O PCU measure. In particular, the attainability constraints allow flexible proportional reduction or expansion for each of the variable inputs, which differs from Kerstens et al. (2019b) that only allow proportional reduction or expansion for all variable inputs by the same ratio. These new constraints add great values to this plant capacity measurement because they are flexible to reflect a variety of scenarios in reality.

This work is structured in the following way. Section 2 defines the technology and reviews some radial and nonradial efficiency measures. In Sect. 3, we define the I-O and O-O radial PCU measures and the partial O-O PCU measure, after which we question the radial efficiency measure in plant capacity measurement. We then propose the new nonradial I-O and O-O PCU measures by a couple of Färe-Lovell efficiency measures. We further explore the attainability issue of the nonradial O-O PCU measure and define an attainable nonradial PCU measure depending on the availability of variable inputs. Section 4 provides an empirical illustration. The contribution ends with some conclusions in Sect. 5.

2 Technology and efficiency measures

In this section we mainly define the technology and review radial and nonradial efficiency measures. Technology T describe all production possibilities transforming input vectors $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m_+$ into output vectors $\boldsymbol{y} = (y_1, \ldots, y_s) \in \mathbb{R}^s_+$, i.e., $T = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{m+s}_+ \mid \boldsymbol{x} \text{ can produce at least } \boldsymbol{y}\}$.

Occasionally, the following standard axioms are imposed on T (see, for example, Hackman (2008)).

Axiom 1: No free lunch and possibility of inaction, i.e., $(\mathbf{0}, \boldsymbol{y}) \in T \Rightarrow \boldsymbol{y} = \mathbf{0}$ and $(\mathbf{0}, \mathbf{0}) \in T$.

Axiom 2: T is a closed subset of $\mathbb{R}_+^{m+s}$.
Axiom 3: Free disposability, i.e., $(\boldsymbol{x}, \boldsymbol{y}) \in T$, $(\boldsymbol{x}', \boldsymbol{y}') \in \mathbb{R}_+^{m+s}$ and $(\boldsymbol{x}, -\boldsymbol{y}) \leq (\boldsymbol{x}', -\boldsymbol{y}') \Rightarrow (\boldsymbol{x}', \boldsymbol{y}') \in T$.
Axiom 4: T is a convex set, i.e., $(\boldsymbol{x}, \boldsymbol{y}) \in T$, $(\boldsymbol{x}', \boldsymbol{y}') \in T$, and $\lambda \in [0, 1] \Rightarrow (\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{x}', \lambda \boldsymbol{y} + (1 - \lambda)\boldsymbol{y}') \in T$.

Note that Axioms 1–3 are conventional regularity conditions within classical production theory, i.e., possibility of inaction, closeness and free disposability of inputs and outputs. The assumption of convexity has been questioned because a wide range of reasons suggest nonconvexities in technology, for example, indivisibilities, economies of specialization and externalities (for details, see Kerstens and Van de Woestyne (2021)). Thus, in our analysis not all the above axioms are simultaneously imposed. Specifically, we discuss both nonconvex and convex technologies by discarding the convexity axiom or not throughout this contribution.

In addition, the output correspondence, which is relative to T, can be defined as $P(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}_+^s \mid \boldsymbol{x}$ can produce at least $\boldsymbol{y}\}$. By analogy, the input correspondence linked with technology T can be depicted by $L(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}_+^m \mid \boldsymbol{x}$ can produce at least $\boldsymbol{y}\}$.

With regard to $P(\boldsymbol{x})$, the following two subsets are important to characterize efficiency measures (Färe & Lovell, 1978). The isoquant of $P(\boldsymbol{x})$ can be defined as

Isoq $P(\boldsymbol{x}) := \{\boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x}), \theta \in (1, \infty) \Rightarrow \theta \boldsymbol{y} \notin P(\boldsymbol{x})\}$ for $\boldsymbol{x} > \boldsymbol{0}$, and Isoq $P(\boldsymbol{0}) \equiv \boldsymbol{0}$. The efficient subset of $P(\boldsymbol{x})$ is

Eff $P(\boldsymbol{x}) := \{\boldsymbol{y} \mid \boldsymbol{y} \in P(\boldsymbol{x}), \forall \boldsymbol{y}' \geq \boldsymbol{y}, \boldsymbol{y}' \neq \boldsymbol{y} \Rightarrow \boldsymbol{y}' \notin P(\boldsymbol{x})\}$ for $\boldsymbol{x} > \boldsymbol{0}$, and Eff $P(\boldsymbol{0}) \equiv \boldsymbol{0}$.

For $L(\boldsymbol{y})$ we can also distinguish between the isoquant on the one hand:

Isoq $L(\boldsymbol{y}) := \{\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y}), \beta \in [0, 1) \Rightarrow \beta \boldsymbol{x} \notin L(\boldsymbol{y})\}$ for $\boldsymbol{y} > \boldsymbol{0}$, and Isoq $L(\boldsymbol{0}) \equiv \boldsymbol{0}$. and the efficient subset on the other hand:

Eff $L(\boldsymbol{y}) := \{\boldsymbol{x} \mid \boldsymbol{x} \in L(\boldsymbol{y}), \forall \boldsymbol{x}' \leq \boldsymbol{x}, \boldsymbol{x}' \neq \boldsymbol{x} \Rightarrow \boldsymbol{x}' \notin L(\boldsymbol{y})\}$ for $\boldsymbol{y} > \boldsymbol{0}$, and Eff $L(\boldsymbol{0}) \equiv \boldsymbol{0}$.

There is potentially a rather large variety of efficiency measures around in the literature. The survey of Russell and Schworm (2009) delves into oriented efficiency models, providing an in-depth analysis of their characteristics and applications. By contrast, the survey by Russell and Schworm (2011) specifically addresses efficiency measures defined relative to the graph of technology, offering valuable insights into this distinct approach to efficiency assessment. Since plant capacity measures have mainly been defined using oriented efficiency models, we ignore efficiency measures defined relative to the graph of technology.[1] In the terminology introduced by Russell and Schworm (2018), we focus on a selection of "path-based indexes" rather than some alternative "slacks-based indexes". In particular, apart from the traditional radial efficiency measures we limit ourselves to the nonradial Färe-Lovell and weighted Färe-Lovell efficiency measures for the purpose of defining new nonradial PCU measures.

The radial input efficiency measure provides a comprehensive characterization of the input correspondence $L(\boldsymbol{y})$:

$$DF_i(\boldsymbol{x}, \boldsymbol{y}) = \min\{\beta \mid \beta \boldsymbol{x} \in L(\boldsymbol{y}), \beta \in [0, 1]\}. \tag{1}$$

The radial input efficiency measure, denoted by $DF_i(\boldsymbol{x}, \boldsymbol{y})$, exhibits the following two properties: (i) it is no greater than unity with (weakly) efficient production on the isoquant of

[1] Kerstens et al. (2020) propose a graph-based PCU measure based on some efficiency measures defined in relation to the graph of technology.

$L(\boldsymbol{y})$, i.e., $DF_i(\boldsymbol{x}, \boldsymbol{y}) \leq 1$ and $DF_i(\boldsymbol{x}, \boldsymbol{y}) = 1 \Longleftrightarrow \boldsymbol{x} \in \text{Isoq } L(\boldsymbol{y})$; and (ii) it exhibits a cost interpretation.

In a similar vein, the radial output efficiency measure provides a comprehensive characterization of the output correspondence $P(\boldsymbol{x})$:

$$DF_o(\boldsymbol{x}, \boldsymbol{y}) = \max\{\theta \mid \theta \boldsymbol{y} \in P(\boldsymbol{x}), \theta \in [1, +\infty)\}. \tag{2}$$

By analogy, two main properties of radial output efficiency measure $DF_o(\boldsymbol{x}, \boldsymbol{y})$ are (i) it is no less than unity with (weakly) efficient production on the isoquant of $P(\boldsymbol{x})$, i.e., $DF_o(\boldsymbol{x}, \boldsymbol{y}) \geq 1$ and $DF_o(\boldsymbol{x}, \boldsymbol{y}) = 1 \Longleftrightarrow \boldsymbol{y} \in \text{Isoq } P(\boldsymbol{x})$; and (ii) it exhibits a revenue interpretation.

Since the radial efficiency measures projects on the isoquant instead of the efficient subset, Färe and Lovell (1978) propose a nonradial Färe-Lovell input efficiency measure that allows for proportional reductions in all input dimensions simultaneously. The measure is defined as:

$$NDF_i(\boldsymbol{x}, \boldsymbol{y}) = \min\{\frac{1}{m}\sum_{i=1}^{m} \beta_i \mid \beta \odot \boldsymbol{x} \in L(\boldsymbol{y}), \beta_i \in [0, 1]\}. \tag{3}$$

where the symbol $\odot$ represents the Hadamard product of two vectors. This nonradial input efficiency measure $NDF_i(\boldsymbol{x}, \boldsymbol{y})$ minimizes the arithmetic mean of dimension-wise reductions in all input dimensions. In particular, its main properties are (i) it is no greater than unity with efficient production on the efficient subset of $L(\boldsymbol{y})$, i.e., $NDF_i(\boldsymbol{x}, \boldsymbol{y}) \leq 1$. Moreover, $NDF_i(\boldsymbol{x}, \boldsymbol{y}) = 1 \Leftrightarrow \boldsymbol{x} \in \text{Eff } L(\boldsymbol{y})$; (ii) strict monotonicity in inputs for given outputs, i.e., $\boldsymbol{x} \in L(\boldsymbol{y}), \boldsymbol{x}' \geq \boldsymbol{x},$ and $\boldsymbol{x}' \neq \boldsymbol{x} \Rightarrow NDF_i(\boldsymbol{x}, \boldsymbol{y}) > NDF_i(\boldsymbol{x}', \boldsymbol{y})$.

Furthermore, Ruggiero and Bretschneider (1998) define a weighted Färe-Lovell input efficiency measure as:

$$WNDF_i(\boldsymbol{x}, \boldsymbol{y}) = \min\{\sum_{i=1}^{m} \eta_i \beta_i \mid \theta \odot \boldsymbol{x} \in L(\boldsymbol{y}), \beta_i \in [0, 1]\}. \tag{4}$$

where $\eta \in \mathbb{R}_+^m$ is the vector whose elements are $\eta_i > 0$ satisfying $\sum_{i=1}^m \eta_i = 1$. This nonradial weighted input efficiency measure $WNDF_i(\boldsymbol{x}, \boldsymbol{y})$ minimizes the weighted sum of dimension-wise reductions in all input dimensions. It offers exactly the same main properties as $NDF_i(\boldsymbol{x}, \boldsymbol{y})$.

By analogy, the nonradial Färe-Lovell output efficiency measure allowing for proportional expansions in all output dimensions simultaneously can be defined as:

$$NDF_o(\boldsymbol{x}, \boldsymbol{y}) = \max\{\frac{1}{s}\sum_{r=1}^{s} \theta_r \mid \theta \odot \boldsymbol{y} \in P(\boldsymbol{x}), \theta_r \in [1, +\infty)\}. \tag{5}$$

It maximizes the arithmetic mean of dimension-wise expansions in all output dimensions. Its main characteristics include (i) it is no less than unity with efficient production on the efficient subset of $P(\boldsymbol{x})$, i.e., $NDF_o(\boldsymbol{x}, \boldsymbol{y}) \geq 1$. Moreover, $NDF_o(\boldsymbol{x}, \boldsymbol{y}) = 1 \Leftrightarrow \boldsymbol{y} \in \text{Eff } P(\boldsymbol{x})$; and (ii) strict monotonicity in outputs when inputs are fixed, i.e., $\boldsymbol{y} \in P(\boldsymbol{x}), \boldsymbol{y}' \leq \boldsymbol{y},$ and $\boldsymbol{y}' \neq \boldsymbol{y} \Rightarrow NDF_o(\boldsymbol{x}, \boldsymbol{y}) < NDF_o(\boldsymbol{x}, \boldsymbol{y}')$.

The corresponding weighted Färe-Lovell output efficiency measure is defined as (Zhu, 1996):

$$WNDF_o(\boldsymbol{x}, \boldsymbol{y}) = \max\{\sum_{r=1}^{s} \mu_r \theta_r \mid \theta \odot \boldsymbol{y} \in P(\boldsymbol{x}), \theta_r \in [1, +\infty)\}. \tag{6}$$

where $\mu \in \mathbb{R}_+^s$ is the vector whose elements are $\mu_r > 0$ satisfying $\sum_{r=1}^s \mu_r = 1$. The nonradial weighted output efficiency measure $WNDF_o(\boldsymbol{x}, \boldsymbol{y})$ maximizes the weighted sum of dimension-wise expansions in all output dimensions. Its main properties are identical to $NDF_o(\boldsymbol{x}, \boldsymbol{y})$.

With regard to the definition of PCU concepts, it is necessary to partition the input vector $\boldsymbol{x}$ into a fixed sub-vector $\boldsymbol{x^f} \in \mathbb{R}_+^{m_f}$ and a variable sub-vector $\boldsymbol{x^v} \in \mathbb{R}_+^{m_v}$ where $m = m_f + m_v$. Following Färe et al. (1989), a short-run technology can be defined as $T^f = \{(\boldsymbol{x^f}, \boldsymbol{y}) \in \mathbb{R}_+^{m_f+s} \mid$ there exists some $\boldsymbol{x^v}$ such that $(\boldsymbol{x^f}, \boldsymbol{x^v})$ can produce at least $\boldsymbol{y}\}$ (see Kerstens et al. (2019a)). Its short-run input correspondence is $L^f(\boldsymbol{y}) = \{\boldsymbol{x^f} \in \mathbb{R}_+^{m_f} \mid (\boldsymbol{x^f}, \boldsymbol{y}) \in T^f\}$, and its short-run output correspondence is $P^f(\boldsymbol{x^f}) = \{\boldsymbol{y} \in \mathbb{R}_+^s \mid (\boldsymbol{x^f}, \boldsymbol{y}) \in T^f\}$. It is evident that technology T^f is a subset of technology T and can be derived by letting all variable inputs $\boldsymbol{x^v}$ be zero. Similarly, $L^f(\boldsymbol{y})$ (resp. $P^f(\boldsymbol{x^f})$) is a subset of $L(\boldsymbol{y})$ (resp. $P(\boldsymbol{x})$) and can be derived by the same projection principles.

Similar to the definitions of Isoq $P(\boldsymbol{x})$, Eff $P(\boldsymbol{x})$, Isoq $L(\boldsymbol{y})$ and Eff $L(\boldsymbol{y})$, we can define the isoquant and efficient subset of the short-run output and input correspondence $P^f(\boldsymbol{x^f})$ and $L^f(\boldsymbol{y})$. To save space, we therefore omit the detailed definition here.

Accordingly, we can define a radial sub-vector input efficiency measure of $L(\boldsymbol{y})$ that only reduces variable inputs proportionally $DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ as:

$$DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) = \min\{\beta \mid (\boldsymbol{x^f}, \beta\boldsymbol{x^v}) \in L(\boldsymbol{y}), \beta \in [0, 1]\}. \tag{7}$$

In addition, the radial output efficiency measure $DF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$ with respect to the short-run output set $P^f(\boldsymbol{x^f})$ is defined as:

$$DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = \max\{\theta \mid \theta\boldsymbol{y} \in P^f(\boldsymbol{x^f}), \theta \in [1, +\infty)\}. \tag{8}$$

Finally, assume there exist n observations consisting of a $(m+s)$-dimensional vector incorporating inputs and outputs $(\boldsymbol{x_j}, \boldsymbol{y_j}) \in \mathbb{R}_+^{m+s}$, $j = 1, ..., n$. Defining nonparametric frontier technologies, a unified mathematical characterization to technology T can be formulated as

$$T^{\Omega} = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \geq \sum_{j=1}^{n} \lambda_j \boldsymbol{x_j}, \boldsymbol{y} \leq \sum_{j=1}^{n} \lambda_j \boldsymbol{y_j}, \lambda \in \Omega\}, \tag{9}$$

where

(i) $\Omega \equiv \Omega^C = \{\lambda \mid \sum_{j=1}^n \lambda_j = 1 \text{ and } \lambda_j \geq 0\}$;
(ii) $\Omega \equiv \Omega^{NC} = \{\lambda \mid \sum_{j=1}^n \lambda_j = 1 \text{ and } \lambda_j \in \{0, 1\}\}$.

Note that the above characterization assumes variable return to scale for simplicity, but it can be extended to a general returns to scale framework (for detail, see Kerstens and Van de Woestyne (2021)). The constraints on activity vector λ identify convex and nonconvex technologies, where all components summing to unity denote the convexity axiom and all component with binary values summing to unity describes nonconvexity. The nonconvex technology is consistent with the free disposable hull developed by Deprins et al. (1984).[2] In the remainder, conventions NC and C are used to represent nonconvex and convex nonparametric frontier technologies, respectively.

[2] There are a variety of other nonconvex technologies, see for example, Petersen (1990), Post (2001), and Podinovski and Kuosmanen (2011). This contribution focuses on the free disposable hull case for ease of exposition.

3 Plant capacity concepts

3.1 Plant capacity: basic definitions

Johansen (1968) initially offers a broad interpretation of plant capacity, defining it as the maximum quantity that can be produced per unit of time using existing plant and equipment, under the condition that the availability of variable factors of production is unrestricted. This definition is intuitive, but it lacks an explicit mathematical characterization in this work. To operationalize this definition, Fare et al. (1989) and Färe et al. (1989) develop an operational O-O PCU measure using nonparametric technologies. This O-O PCU measure is defined as follows.

Definition 3.1 The O-O PCU PCU_o is defined as:

$$PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) = \frac{DF_o(\boldsymbol{x}, \boldsymbol{y})}{DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})}, \tag{10}$$

where $DF_o(\boldsymbol{x}, \boldsymbol{y})$ and $DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$ are radial output efficiency measures using production correspondences $P(\boldsymbol{x})$ and $P^f(\boldsymbol{x}^f)$, respectively.

Note that $0 < PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) \leq 1$, since $1 \leq DF_o(\boldsymbol{x}, \boldsymbol{y}) \leq DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$. In particular, $PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) = 1$ implies that the current combination of variable and fixed inputs is sufficient to generate the maximum capacity outputs. Otherwise if $PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) < 1$, then the maximum capacity outputs cannot be reached because of the limited current variable inputs.[3] Taking a closer look at the PCU measure, Färe et al. (1994) denote $DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$ and $PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$ as a biased and an unbiased PCU measure, respectively.

To further explore the capacity utilization level of a specific output, a partial O-O PCU measure, where only one output is variable while all other outputs are held fixed, is defined by Vestergaard et al. (2003). This partial O-O PCU measure is defined as:

Definition 3.2 The partial O-O PCU of output r, $DF_{o(r)}^f$, $r = 1, ..., s$ is defined as

$$DF_{o(r)}^f(\boldsymbol{x}^f, y_r, \boldsymbol{y_{-r}}) = \max\{\theta_r \mid (\theta_r y_r, \boldsymbol{y_{-r}}) \in P^f(\boldsymbol{x}^f), \theta_r \in [1, +\infty)\}, \tag{11}$$

where y_r denotes the rth component of output vector $\boldsymbol{y}$ and $\boldsymbol{y_{-r}}$ denotes the sub-vector of output excluding y_r.

Different from $PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$, this new measure is derived from a nonradial efficiency measure, which allows proportional expansion of the rth output only. $DF_{o(r)}^f(\boldsymbol{x}^f, y_r, \boldsymbol{y_{-r}}) \geq 1$ always holds by definition and the equality indicates the rth output has achieved its maximal capacity when the fixed inputs are completely used and all other outputs maintain current amounts. Particularly, following the terminology by Färe et al. (1994), $DF_{o(r)}^f(\boldsymbol{x}^f, y_r, \boldsymbol{y_{-r}})$ denotes a biased PCU measure rather than an unbiased one.

To our knowledge, an unbiased measure to the partial O-O PCU has not been defined explicitly, therefore we define it here.

[3] Note that $PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) = 1$ does not imply that $(\boldsymbol{x}, \boldsymbol{y})$ is efficient because the efficiency status has no impact on plant capacity measurement (Cesaroni et al. (2017)).

Definition 3.3 The unbiased measure to the partial O-O PCU of output r ($PPCU_{o(r)}$), $r = 1, ..., s$ is defined as

$$PPCU_{o(r)}(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, y_r, \boldsymbol{y_{-r}}) = \frac{DF_{o(r)}(\boldsymbol{x}, y_r, \boldsymbol{y_{-r}})}{DF^{f}_{o(r)}(\boldsymbol{x}^{\boldsymbol{f}}, y_r, \boldsymbol{y_{-r}})} \tag{12}$$

where $DF_{o(r)}(\boldsymbol{x}, y_r, \boldsymbol{y_{-r}}) = \max\{\theta_r \mid (\theta_r y_r, \boldsymbol{y_{-r}}) \in P(\boldsymbol{x}), \theta_r \geq 0\}$. Note that as $1 \leq DF_{o(r)}(\boldsymbol{x}, y_r, \boldsymbol{y_{-r}}) \leq DF^{f}_{o(r)}(\boldsymbol{x}^{\boldsymbol{f}}, y_r, \boldsymbol{y_{-r}})$, $0 < PPCU_{o(r)}(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, y_r, \boldsymbol{y_{-r}}) \leq 1$ holds, i.e., $PPCU_{o(r)}(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, y_r, \boldsymbol{y_{-r}})$ has an upper bound of unity. In particular, $PPCU_{o(r)}(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, y_r, \boldsymbol{y_{-r}}) = 1$ indicates the current variable inputs, together with the current fixed inputs, are enough to produce the maximal capacity of the rth output; otherwise, the maximal capacity of the rth output is still limited by the current variable inputs.

In addition to the O-O PCU measure, an I-O PCU measure consisting of the ratio of two I-O efficiency measures is first explored by Cesaroni et al. (2017).

Definition 3.4 The I-O PCU PCU_i is defined as:

$$PCU_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \boldsymbol{y}) = \frac{DF^{SR}_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \boldsymbol{y})}{DF^{SR}_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \epsilon)}, \tag{13}$$

where ϵ is a non-Archimedean infinitesimal and $DF^{SR}_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \boldsymbol{y})$ and $DF^{SR}_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \epsilon)$ are sub-vector radial input efficiency measures using input correspondences $L(\boldsymbol{y})$ and $L(\epsilon)$, respectively.

Since $0 < DF^{SR}_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \epsilon) \leq DF^{SR}_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \boldsymbol{y})$, $PCU_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \boldsymbol{y}) \geq 1$ holds. In particular, $PCU_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \boldsymbol{y}) = 1$ implies that the maximal proportional reduction of variable inputs to produce the current outputs is equal to that to produce ϵ outputs. By analogy, we can differentiate between a biased I-O PCU measure denoted by $DF^{SR}_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \epsilon)$ and an unbiased I-O PCU measure denoted by $PCU_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \boldsymbol{y})$.

3.2 Plant capacity: questioning radial measures

Although the Definitions 3.1 and 3.4 are sufficiently well-defined to measure PCU in practice, we emphasize that these measures may be still biased in term of the usage of radial efficiency measures.[4] From an efficiency perspective, radial efficiency measures cannot completely capture inefficiencies in all dimensions. As a result, these plant capacity measures may not fully capture the optimal capacity of (variable) inputs or the optimal capacity of outputs in the plant capacity concept.

For ease of exposition, it is useful to introduce the following definitions.

Definition 3.5 The maximal output capacity from the O-O PCU $\boldsymbol{y}_{o,(\boldsymbol{x}^{\boldsymbol{f}},\boldsymbol{y})}$ is defined as

$$\boldsymbol{y}_{o,(\boldsymbol{x}^{\boldsymbol{f}},\boldsymbol{y})} = DF^{f}_o(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y})\, \boldsymbol{y}. \tag{14}$$

Definition 3.6 The minimal variable input capacity from the I-O PCU $\boldsymbol{x^v}_{i,(\boldsymbol{x}^{\boldsymbol{f}},\boldsymbol{x}^{\boldsymbol{v}},\epsilon)}$ is defined as

$$\boldsymbol{x^v}_{i,(\boldsymbol{x}^{\boldsymbol{f}},\boldsymbol{x}^{\boldsymbol{v}},\epsilon)} = DF^{SR}_i(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{x}^{\boldsymbol{v}}, \epsilon)\boldsymbol{x}^{\boldsymbol{v}}. \tag{15}$$

[4] The partial O-O PCU $PPCU_{o(r)}(\boldsymbol{x}^{\boldsymbol{f}}, y_r, \boldsymbol{y_{-r}})$ from Definition 3.3 is defined by nonradial efficiency measures: we discuss it later in more detail.

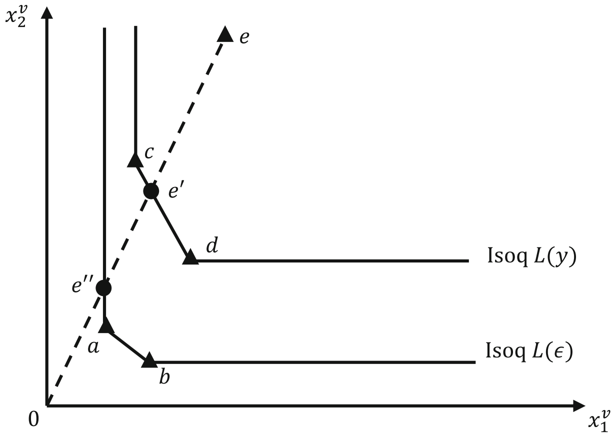


Fig. 1 Overestimation of the optimal capacity of variable inputs with I-O PCU measure

Note that the maximal output (resp. minimal input) capacity are denoted by the biased PCU measure times current outputs (resp. inputs). As $DF_o^f(\boldsymbol{x^f}, y) \geq 1$ and $DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) \leq 1$, $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \geq \boldsymbol{y}$ and $\boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)} \leq \boldsymbol{x^v}$ always hold. The potential increment on outputs and decrement on variable inputs are described by $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} - \boldsymbol{y}$ and $\boldsymbol{x^v} - \boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}$. Such potentials are denoted as *slacks* in the context of production economics (see, for example, Cooper et al. (1999) and Tone (2001)).

In addition, by using radial efficiency measures, we can state the results for the maximal output capacity as well as the corresponding isoquants.

Proposition 3.1 *The maximal output capacity* $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ *has the following properties:*

(i) It belongs to the isoquant of $P^f(\boldsymbol{x^f})$, *i.e.,* $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ *Isoq* $P^f(\boldsymbol{x^f})$.

(ii) It belongs to the isoquant of $P(\boldsymbol{x^f}, +\infty)$, *i.e.,* $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ *Isoq* $P(\boldsymbol{x^f}, +\infty)$.

Proposition 3.2 *The minimal input capacity* $\boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}$ *with the fixed inputs* $\boldsymbol{x^f}$ *belongs to the isoquant of* $L(\epsilon)$, *i.e.,* $(\boldsymbol{x^f}, \boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}) \in$ *Isoq* $L(\epsilon)$.

The proofs of all propositions are provided in the Appendix A.

We next argue the optimality of $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ and $\boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}$ in plant capacity measurement. As shown in Propositions 3.1 and 3.2, $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ and $\boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}$ is related to isoquants Isoq $P^f(\boldsymbol{x^f})$, Isoq $P(\boldsymbol{x})$, and Isoq $L(\epsilon)$. Note that the isoquants cannot capture all inefficiencies because they are defined by radial efficiency measure. This property inevitably has impact on the maximal output capacity and minimal input capacity, which may lead to inaccurate O-O and I-O PCU measures. To illustrate, we argue the optimality of $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ and $\boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}$ with the help of Figures 1 and 2. The observations are denoted by triangles and projections by circles.

Given specific fixed inputs, Fig. 1 depicts the overestimation of the optimal capacity of variable inputs and elucidates its impacts on the I-O PCU measure in a space of two variable inputs (x_1^v, x_2^v). The isoquant of $L(\boldsymbol{y})$ is the line cd, its horizonal extension at d, and vertical extension at c. The line ab, its horizonal extension at b, and vertical extension at a describe the isoquant of $L(\epsilon)$. We focus on observation e. The I-O PCU measure compares observation e to its radial projection e' on the segment cd of Isoq $L(\boldsymbol{y})$, i.e., $DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) = \frac{\|oe'\|}{\|oe\|}$.

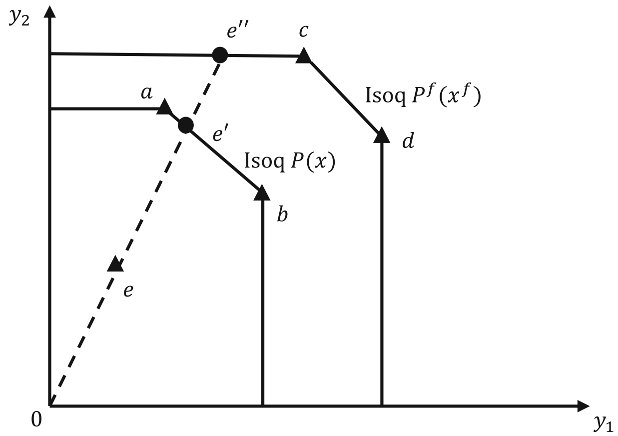


Fig. 2 Underestimation of the optimal capacity of outputs with O-O PCU measure

It also compares observation e to its radial projection e'' on the vertical extension of segment ab at a of Isoq $L(\epsilon)$, i.e., $DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) = \frac{\|oe''\|}{\|oe\|}$. Consequently, the I-O PCU measure is denoted as $PCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) = \frac{DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})}{DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)} = \frac{\|oe'\|}{\|oe''\|}(> 1)$. In particular, the minimal input capacity from I-O PCU is denoted by projection e''. Clearly, the minimal input capacity is not optimal although it belongs to Isoq $L(\epsilon)$ because of the ignored slacks in variable input x_2^v (denoted by distance $\|ae''\|$). Compared with the minimal input capacity e'', observation a uses less variable input x_2^v but produces identical outputs ϵ. As a consequence, the projection e'' overestimates the optimal capacity of variable input x_2^v, where the optimal capacity of variable inputs (x_1^v, x_2^v) should be denoted as observation a. The overestimated input capacity is not embodied in Definition 3.4, which may lead to inaccurate I-O plant capacity measurement.

By analogy, given specific (fixed and variable) inputs, Fig. 2 visualizes the underestimation of the optimal capacity of outputs and illustrates its impacts on the O-O PCU measure in a two outputs (y_1, y_2) space. The isoquant of $P(\boldsymbol{x})$ is the line ab, its horizonal extension at a, and vertical extension at b. The line cd, its horizonal extension at c and vertical extension at d constitute the isoquant of $P^f(\boldsymbol{x^f})$. We use observation e for illustration.[5] The O-O PCU measure compares observation e to its radial projections e' on the segment ab of Isoq $P(\boldsymbol{x})$, i.e., $DF_o(\boldsymbol{x}, \boldsymbol{y}) = \frac{\|oe'\|}{\|oe\|}$. It also compares observation e to its radial projection e'' on the horizonal extension of segment cd at c of Isoq $P^f(\boldsymbol{x^f})$, i.e., $DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = \frac{\|oe''\|}{\|oe\|}$. Accordingly, the O-O PCU measure is denoted as $PCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}) = \frac{DF_o(\boldsymbol{x}, \boldsymbol{y})}{DF_o^f(\boldsymbol{x^f}, \boldsymbol{y})} = \frac{\|oe'\|}{\|oe''\|}(< 1)$. Particularly, the maximal output capacity from O-O PCU is denoted by projection e''. It is obvious that the maximal output capacity is not optimal although it belongs to Isoq $P^f(\boldsymbol{x^f})$ due to the slacks in output y_1 (denoted by distance $\|e''c\|$). Compared with the maximal output capacity e'', observation c consumes identical fixed inputs $(\boldsymbol{x^f})$ but generates more output y_1. As a consequence, the radial projection e'' underestimates the optimal capacity of output y_1, where the optimal capacity of outputs (y_1, y_2) should be denoted as observation c. Such underestimated output capacity is not captured in Definition 3.1, thereby resulting in incorrect O-O plant capacity measurement.

[5] The observations and projections in Fig. 2 differ from those in Fig. 1 although we use the same letters.

Wrapping up, the biased estimations on the optimal input/output capacity question the practicality of radial efficiency measures in plant capacity measurement. It is useful to stress that the biases disappear when output is a singleton (i.e., $s = 1$) in O-O PCU measure, and when variable input is a singleton (i.e., $m^v = 1$) in I-O PCU measures. Therefore, Definitions 3.1 and 3.4 are theoretically applicable in scenarios involving either single variable input or single output cases. However, they may be biased for the production processes considering multiple variable inputs or multiple outputs. To solve this question, nonradial efficiency measures are introduced to define new nonradial plant capacity measures based on optimal input and output capacities.[6] The widely used traditional radial efficiency measures often overlook the complexities inherent in multi-dimensional input and output structures. These measures assume proportional adjustments across all dimensions, which may not reflect real-world production processes where inputs and outputs can be adjusted in a non-uniform way. By contrast, the nonradial models allow for non-proportional adjustments, enabling a more nuanced evaluation of production performance.

In the next subsection, we propose a nonradial PCU framework that leverages Färe-Lovell efficiency measures. This framework is designed to comprehensively capture multidimensional inefficiencies and incorporate realistic operational constraints, thereby providing a more accurate and flexible assessment of production capacities.

3.3 Nonradial plant capacity measures: proposals

We now proceed to outline the formulation of nonradial plant capacity measures using the Färe-Lovell efficiency measures. We first define a Färe-Lovell O-O PCU measure. Thereafter, we explore how to define a Färe-Lovell I-O PCU measure.

3.3.1 Nonradial output-oriented plant capacity measure

To begin with, we first define the biased Färe-Lovell O-O PCU measure utilizing the Färe-Lovell output efficiency measure as

$$NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y}) = \max\{\frac{1}{s}\sum_{r=1}^{s}\theta_r \mid \theta \odot \boldsymbol{y} \in P^f(\boldsymbol{x}^f), \theta_r \in [1, +\infty)\}. \tag{16}$$

Accordingly, we can define an unbiased Färe-Lovell O-O PCU measure $NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$ involving two Färe-Lovell output efficiency measures in relation to both production correspondence $P(\boldsymbol{x})$ and the same correspondence exposing no restrictions on variable inputs $P^f(\boldsymbol{x}^f)$.

Definition 3.7 The Färe-Lovell O-O PCU $NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$ is defined as:

$$NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) = \frac{NDF_o(\boldsymbol{x}, \boldsymbol{y})}{NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})}, \tag{17}$$

where $NDF_o(\boldsymbol{x}, \boldsymbol{y})$ and $NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$ are Färe-Lovell O-O efficiency measures related to production correspondences including and excluding variable inputs, respectively.

Since $0 < NDF_o(\boldsymbol{x}, \boldsymbol{y}) \leq NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$, $0 < NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) \leq 1$. Thus, the Färe-Lovell O-O PCU measure attains a maximum value of unity.

[6] The optimal input capacity should be located on segment ab with Fig. 1, and the optimal output capacity should be located on segment cd with Fig. 2.

The Färe-Lovell output efficiency measures relative to production correspondence $P(\boldsymbol{x})$ ($NDF_o(\boldsymbol{x}, \boldsymbol{y})$) can be obtained by solving the linear program (B.3) in Appendix B. The Färe-Lovell output efficiency measure relative to production correspondence $P^f(\boldsymbol{x}^f)$ ($NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$) is computed as the linear program (B.4) in Appendix B.

Definition 3.7 gives rise to the O-O decomposition below:

$$NDF_o(\boldsymbol{x}, \boldsymbol{y}) = NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) \cdot NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y}). \tag{18}$$

As a consequence, the Färe-Lovell output efficiency measure $NDF_o(\boldsymbol{x}, \boldsymbol{y})$ can be decomposed into a biased Färe-Lovell O-O PCU measure $NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$ and an unbiased Färe-Lovell O-O PCU measure $NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$.

In addition, a new definition of weighted Färe-Lovell O-O PCU measure $WNPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$ involves two weighted Färe-Lovell output efficiency measures related to both output sets $P(\boldsymbol{x})$ and the same correspondence assuming no restrictions on variable inputs $P^f(\boldsymbol{x}^f)$, respectively. The weighted Färe-Lovell output efficiency measure relative to production correspondence $P^f(\boldsymbol{x}^f)$ is defined as

$$WNDF_o^f(\boldsymbol{x}^f, \boldsymbol{y}) = \max\{\sum_{r=1}^{s} \mu_r \theta_r \mid \theta \odot \boldsymbol{y} \in P^f(\boldsymbol{x}^f), \theta_r \in [1, +\infty)\}. \tag{19}$$

We are now ready to define a weighted Färe-Lovell O-O PCU measure.

Definition 3.8 The weighted Färe-Lovell O-O PCU $WNPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$ is defined as

$$WNPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) = \frac{WNDF_o(\boldsymbol{x}, \boldsymbol{y})}{WNDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})}, \tag{20}$$

where $WNDF_o(\boldsymbol{x}, \boldsymbol{y})$ and $WNDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$ are weighted Färe-Lovell O-O efficiency measures relative to production correspondences including and ignoring variable inputs, respectively.

Note that since $0 < WNDF_o(\boldsymbol{x}, \boldsymbol{y}) \leq WNDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$, then $0 < WNPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) \leq 1$. Hence, the weighted Färe-Lovell O-O PCU attains a maximum value of unity. This weighted Färe-Lovell O-O PCU measure evaluates the maximal weighted sum of dimension-wise expansions in all output dimensions in the sample against the maximal weighted sum of dimension-wise expansions in outputs with potentially unlimited variable inputs.

Referring to nonparametric frontier technologies, we can obtain the weighted Färe-Lovell output efficiency measure relative to production correspondence $P(\boldsymbol{x})$ ($WNDF_o(\boldsymbol{x}, \boldsymbol{y})$) using the linear program (B.1) in Appendix B. The weighted Färe-Lovell output efficiency measure relative to production correspondence $P^f(\boldsymbol{x}^f)$ ($WNDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$) for observation $(\boldsymbol{x}_k, \boldsymbol{y}_k)$, $k = 1, ..., n$ is computed as the linear program (B.2) in Appendix B.

In particular, Definition 3.8 gives rise to the O-O decomposition below:

$$WNDF_o(\boldsymbol{x}, \boldsymbol{y}) = WNPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}) \cdot WNDF_o^f(\boldsymbol{x}^f, \boldsymbol{y}). \tag{21}$$

Therefore, the weighted Färe-Lovell output efficiency measure $WNDF_o(\boldsymbol{x}, \boldsymbol{y})$ can be decomposed into a biased weighted Färe-Lovell O-O PCU measure $WNDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$ and an unbiased weighted Färe-Lovell O-O PCU measure $WNPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$.

Accordingly, that the corresponding optimal output capacity can be defined.

Definition 3.9 The optimal output capacity from the weighted Färe-Lovell O-O PCU $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ is defined as:

$$\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} = \theta^* \odot \boldsymbol{y}, \tag{22}$$

where $\theta^* \in \mathbb{R}^r_+$ whose elements are $\theta^*_r \geq 1$ is the optimal solution of (19). The optimal output capacity is denoted by the element-by-elment operands of two s-dimensional vectors θ^* relative to the weighted Färe-Lovell O-O PCU measure and $\boldsymbol{y}$. As $\theta^* \geq 1$, $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \geq \boldsymbol{y}$ always holds. In addition, by using such nonradial efficiency measure, we can state the result for the optimal output capacity as well as the corresponding efficient subsets.

Proposition 3.3 *The optimal output capacity $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ has the following properties:*

(i) It pertains to the efficient subset of $P^f(\boldsymbol{x}^f)$, i.e., $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in Eff\ P^f(\boldsymbol{x}^f)$.
(ii) It pertains to the efficient subset of $P(\boldsymbol{x}^f,+\infty)$, i.e., $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in Eff\ P(\boldsymbol{x}^f,+\infty)$.

Proposition 3.3 indicates the optimal output capacity $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ from nonradial efficiency measures is located on the efficient subsets of $P^f(\boldsymbol{x}^f)$ and $P(\boldsymbol{x}^f,+\infty)$ rather than on the isoquants from radial efficiency measures in Proposition 3.1. These properties ensure that all *slacks* in outputs are completely captured in this nonradial O-O PCU measure.

With regard to the new nonradial O-O PCU measure, we can now develop a generalized framework for the biased O-O PCU measure whereby radial and nonradial O-O PCU measures are its special cases. This result is as follows.

Proposition 3.4 *The generalized framework for the biased O-O PCU measure is defined as:*

$$GDF^f_o(\boldsymbol{x}^f,\boldsymbol{y} \mid \Lambda,\Gamma) = \max\{\sum_{r=1}^{s}\mu_r\theta_r \mid \theta\odot\boldsymbol{y} \in P^f(\boldsymbol{x}^f), \mu\in\Lambda, \theta\in\Gamma\}. \tag{23}$$

whereby:

(i) $\Lambda = \Lambda^1 = \{\mu \mid \mu_1 = \mu_2 = \cdots = \mu_s = \frac{1}{s}\}$, $\Gamma = \Gamma^1 = \{\theta \mid \theta_1 = \theta_2 = \cdots = \theta_s \geq 1\} \Rightarrow GDF^f_o(\boldsymbol{x}^f,\boldsymbol{y} \mid \Lambda,\Gamma) = DF^f_o(\boldsymbol{x}^f,\boldsymbol{y})$;
(ii) $\Lambda = \Lambda^2 = \{\mu \mid \mu_r = 1, \mu_{-r} = 0\}$, $\Gamma = \Gamma^2 = \{\theta \mid \theta_r \geq 1, \theta_{-r} = 1\} \Rightarrow GDF^f_o(\boldsymbol{x}^f,\boldsymbol{y} \mid \Lambda,\Gamma) = DF^f_{o(r)}(\boldsymbol{x}^f, y_r, \boldsymbol{y_{-r}})$;
(iii) $\Lambda = \Lambda^3 = \{\mu \mid \sum_{r=1}^{s}\mu_r = 1, \mu_r > 0, r = 1,...,s\}$, $\Gamma = \Gamma^3 = \{\theta \mid \theta \geq 1\} \Rightarrow GDF^f_o(\boldsymbol{x}^f,\boldsymbol{y} \mid \Lambda,\Gamma) = WNDF^f_o(\boldsymbol{x}^f,\boldsymbol{y})$;
(iv) $\Lambda = \Lambda^1 = \{\mu \mid \mu_1 = \mu_2 = \cdots = \mu_s = \frac{1}{s}\}$, $\Gamma = \Gamma^3 = \{\theta \mid \theta \geq 1\} \Rightarrow GDF^f_o(\boldsymbol{x}^f,\boldsymbol{y} \mid \Lambda,\Gamma) = NDF^f_o(\boldsymbol{x}^f,\boldsymbol{y})$;

Note that μ is a prior vector rather than a variable vector in the proposed generalized framework. Proposition 3.4 indicates that the biased radial O-O PCU measure $DF^f_o(\boldsymbol{x}^f,\boldsymbol{y})$ and the nonradial O-O PCU measures $DF^f_{o(r)}(\boldsymbol{x}^f, y_r, \boldsymbol{y_{-r}})$ and $WNDF^f_o(\boldsymbol{x}^f,\boldsymbol{y})$ can be integrated into one generalized framework. In particular, the biased radial O-O PCU measure is a special case of the biased weighted Färe-Lovell O-O PCU measure because of $\Lambda^1 \subset \Lambda^3$ and $\Gamma^1 \subset \Gamma^3$. Specifically, the biased radial O-O PCU measure assigns same values to the components of vectors μ and θ in tandem.

With regard to the generalized framework and its special cases, we are now in a position to identify the linkages between biased radial and nonradial O-O PCU measures.

Proposition 3.5 *The following linkages can be established among biased radial O-O PCU measure, partial O-O PCU measure, and Färe-Lovell O-O PCU measure ($s \geq 1$):*

$$1 \leq DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) \leq DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) \leq NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}), r = 1, ..., s. \quad (24)$$

In particular,

(i) a sufficient condition for $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) < NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}), r = 1, ..., s$ is that $\boldsymbol{y} \notin Eff\ P^f(\boldsymbol{x^f})$, i.e., $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) > 1$;

(ii) a sufficient condition for $DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})$ is that output is a singleton, i.e., $s = 1$.

Proposition 3.5 shows that the biased radial O-O PCU measure is no more than the two biased nonradial O-O PCU measures. This property is important because it implies the result that plant capacity measurement using the radial output efficiency measure may be biased downwards. In particular, when the outputs are not located on the efficient subset of the output set $P^f(\boldsymbol{x^f})$, then the partial O-O PCU measure is less than the Färe-Lovell O-O PCU measure. This result indicates that the partial O-O PCU measure is strictly biased downward for inefficient observations in plant capacity measurement. Proposition 3.5 also reports the equivalence property of the three biased O-O PCU measure and its sufficient condition. The equivalence property shows that the biased radial O-O PCU measure and partial O-O PCU measure are unbiased estimations of the biased Färe-Lovell O-O PCU measure when the output is a singleton.

By contrast, such linkages are not established for the unbiased radial and nonradial O-O PCU measures. Although both numerators ($1 \leq DF_o(\boldsymbol{x}, \boldsymbol{y}) \leq DF_{o(r)}^f(\boldsymbol{x}, y_r, \boldsymbol{y_{-r}}) \leq NDF_o(\boldsymbol{x}, \boldsymbol{y}), r = 1, ..., s$) and denominators ($1 \leq DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) \leq DF_{o(r)}(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) \leq NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}),\ r = 1, ..., s$) can be ranked completely, the ratios of them ($PCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$, $PPCU_{o(r)}(\boldsymbol{x}, \boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})$ and $NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$) can not be ranked.[7]

3.3.2 Nonradial input-oriented plant capacity measure

We first define a new measure, called the biased Färe-Lovell I-O PCU measure, referring to the Färe-Lovell input efficiency measure relative to input set $L(\epsilon)$ as:

$$NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) = \min\{\frac{1}{m^v}\sum_{i=1}^{m^v} \beta_i \mid (\boldsymbol{x^f}, \beta \odot \boldsymbol{x^v}) \in L(\epsilon), \beta_i \in [0, 1]\}. \quad (25)$$

The sub-vector Färe-Lovell input efficiency measure relative to input set $L(\boldsymbol{y})$ can be defined as:

$$NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) = \min\{\frac{1}{m^v}\sum_{i=1}^{m^v} \beta_i \mid (\boldsymbol{x^f}, \beta \odot \boldsymbol{x^v}) \in L(\boldsymbol{y}), \beta_i \in [0, 1]\}. \quad (26)$$

Accordingly, we can define an unbiased Färe-Lovell I-O PCU measure $NPCU_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ involving two Färe-Lovell input efficiency measures relative to input sets $L(\boldsymbol{y})$ and $L(\epsilon)$.

[7] $1 \leq DF_o(\boldsymbol{x}\boldsymbol{y}) \leq DF_{o(r)}(\boldsymbol{x}, y_r, \boldsymbol{y_{-r}}) \leq NDF_o(\boldsymbol{x}, \boldsymbol{y}), r = 1, ..., s$ can be proven analogous to the proof of Proposition 3.5. The corresponding proof is not provided to save space.

Definition 3.10 The Färe-Lovell I-O PCU $NPCU_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ is defined as

$$NPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) = \frac{NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})}{NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)}. \tag{27}$$

Since $NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) \geq NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$, $NPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) \geq 1$. Therefore, the Färe-Lovell I-O PCU measure attains a minimum value of unity. But, it does not possess a maximum limit.

To compute the sub-vector Färe-Lovell input efficiency measure relative to input set $L(\boldsymbol{y})$ ($NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$) one needs to solve the linear program (B.7) in Appendix B. The biased Färe-Lovell I-O PCU measure in relation to input set $L(\epsilon)$ ($NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$) can be obtained by calculating the linear program (B.8) in Appendix B.

In addition, a new definition of a weighted Färe-Lovell I-O PCU measure $WNPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ involves two sub-vector weighted Färe-Lovell input efficiency measures relative to both input correspondences $L(\boldsymbol{y})$ and $L(\epsilon)$, respectively. The sub-vector weighted Färe-Lovell input efficiency measure in relation to the input set $L(\boldsymbol{y})$ that only reduces variable inputs can be defined as:

$$WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) = \min\{\sum_{i=1}^{m^v} \eta_i \beta_i \mid (\boldsymbol{x^f}, \beta \odot \boldsymbol{x^v}) \in L(\boldsymbol{y}), \beta_i \in [0, 1]\}. \tag{28}$$

Next, the sub-vector weighted Färe-Lovell input efficiency measure in relation to the input set $L(\epsilon)$ that reduces variable inputs only is defined as:

$$WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) = \min\{\sum_{i=1}^{m^v} \eta_i \beta_i \mid (\boldsymbol{x^f}, \beta \odot \boldsymbol{x^v}) \in L(\epsilon), \beta_i \in [0, 1]\}. \tag{29}$$

Now we are in a position to define a weighted Färe-Lovell I-O PCU measure.

Definition 3.11 The weighted Färe-Lovell I-O PCU $WNPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ is defined as

$$WNPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) = \frac{WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})}{WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)}. \tag{30}$$

where $WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ and $WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$ are sub-vector weighted Färe-Lovell input efficiency measures relative to input correspondences with $\boldsymbol{y}$ and ϵ outputs levels, respectively.

Note that $0 < WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) \leq WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ due to $\epsilon < \boldsymbol{y}$, $WNPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) \geq 1$. Therefore, the weighted Färe-Lovell I-O PCU measure attains a minimum value of unity. But, it does not possess a maximum limit. This PCU measure compares the minimal weighted sum of dimension-wise reductions in all variable inputs dimensions in the sample to the minimal weighted sum of dimension-wise reductions in all variable inputs dimensions with an ϵ output level.

One can obtain the sub-vector weighted Färe-Lovell input efficiency measure relative to input correspondence $L(\boldsymbol{y})$ ($WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$) using the linear program (B.5) in Appendix B. The sub-vector weighted Färe-Lovell input efficiency measure relative to input correspondence $L(\epsilon)$ ($WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$) is computed as the linear program (B.6) in Appendix B.

In addition, Definition 3.11 leads to the following I-O decomposition:

$$WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) = WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) \cdot WNPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}). \quad (31)$$

Thus, the sub-vector weighted Färe-Lovell input efficiency measure $WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ can be decomposed into a biased weighted Färe-Lovell I-O PCU measure $WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$ and an unbiased weighted Färe-Lovell I-O PCU measure $WNPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$.

Accordingly, the corresponding optimal input capacity can be defined.

Definition 3.12 The optimal input capacity from the weighted Färe-Lovell I-O PCU ($\boldsymbol{x}^{v,WN}_{i,(\boldsymbol{x^f},\boldsymbol{x^v},\epsilon)}$) is defined as:

$$\boldsymbol{x}^{v,WN}_{i,(\boldsymbol{x^f},\boldsymbol{x^v},\epsilon)} = \beta^* \odot \boldsymbol{x^v}, \quad (32)$$

where $\beta^* \in \mathbb{R}_+^{m^v}$ whose elements are $\beta_i^* > 0$ is the optimal solution of (29). The optimal input capacity is denoted by the element-by-element operands of two m^v-dimensional vectors β^* relative to the weighted Färe-Lovell I-O PCU measure and y. Since $\beta^* \leq 1$, $\boldsymbol{x}^{v,WN}_{i,(\boldsymbol{x^f},\boldsymbol{x^v},\epsilon)} \leq \boldsymbol{x^v}$ always holds.

Moreover, by using a nonradial input efficiency measure, the following result for the optimal input capacity as well as the corresponding isoquant can be reported.

Proposition 3.6 *The optimal input capacity* $\boldsymbol{x}^{v,WN}_{i,(\boldsymbol{x^f},\boldsymbol{x^v},\epsilon)}$ *with the fixed inputs* $\boldsymbol{x^f}$ *belongs to the isoquant of* $L(\epsilon)$*, i.e.,* $(\boldsymbol{x^f}, \boldsymbol{x}^{v,WN}_{i,(\boldsymbol{x^f},\boldsymbol{x^v},\epsilon)}) \in Isoq\ L(\epsilon)$.

Proposition 3.6 indicates that the optimal input capacity $\boldsymbol{x}^{v,WN}_{i,(\boldsymbol{x^f},\boldsymbol{x^v},\epsilon)}$ with the fixed inputs $\boldsymbol{x^f}$ from nonradial efficiency measure is merely located on the isoquant of $L(\epsilon)$. Different from the optimal output capacity, it does not belong to the corresponding efficient subset Eff $L(\epsilon)$ because the optimal input capacity only eliminates all inefficiencies in variable inputs $\boldsymbol{x^v}$ but the *slacks* in fixed inputs $\boldsymbol{x^f}$ are ignored. However, the optimal input capacity deals with the underestimation issue as shown in Fig. 1 since all *slacks* in variable inputs are completely captured in I-O PCU measure.

With regard to the new nonradial I-O PCU measure, we now can propose a generalized framework for the biased I-O PCU measure where both radial and nonradial I-O PCU measures are special cases.

Proposition 3.7 *The generalized framework for the biased I-O PCU measure is defined as:*

$$GDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon \mid \Upsilon, \Phi) = \min\{\sum_{i=1}^{m^v} \eta_i \beta_i \mid (\boldsymbol{x^f}, \beta \odot \boldsymbol{x^v}) \in L(\epsilon), \eta \in \Upsilon, \beta \in \Phi\}, \quad (33)$$

where

(i) $\Upsilon = \Upsilon^1 = \{\eta \mid \eta_1 = \eta_2 = \cdots = \eta_{m^v} = \frac{1}{m^v}\}$, $\Phi = \Phi^1 = \{\beta \mid 0 \leq \beta_1 = \beta_2 = \cdots = \beta_{m^v} \leq 1\} \Rightarrow GDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon \mid \Upsilon, \Phi) = DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$;

(ii) $\Upsilon = \Upsilon^2 = \{\eta \mid \sum_{i=1}^{m^v} \eta_i = 1, \eta_i > 0, i = 1, ..., m^v\}$, $\Phi = \Phi^2 = \{\beta \mid 0 \leq \beta \leq 1\} \Rightarrow GDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon \mid \Upsilon, \Phi) = WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$.

(iii) $\Upsilon = \Upsilon^1 = \{\eta \mid \eta_1 = \eta_2 = \cdots = \eta_{m^v} = \frac{1}{m^v}\}, \Phi = \Phi^2 = \{\beta \mid 0 \leq \beta \leq 1\} \Rightarrow GDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon \mid \Upsilon, \Phi) = NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$.

Note that η is a prior vector instead of a variable vector in the proposed framework. Proposition 3.7 reports that the biased (sub-vector) radial I-O PCU measure $DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$ and nonradial I-O PCU measure $WNDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$ can be integrated into a generalized framework. In particular, the biased radial I-O PCU measure is a special case of the biased weighted Färe-Lovell I-O PCU measure due to $\Upsilon^1 \subset \Upsilon^2$ and $\Phi^1 \subset \Phi^2$. The biased (sub-vector) radial I-O PCU measure assigns the same values to the components of vector η and β.

With regard to the generalized framework and its special cases, we now establish a linkage between the biased (sub-vector) radial and nonradial I-O PCU measures.

Proposition 3.8 *The following linkage can be established between the biased (sub-vector) radial I-O PCU measure and the biased Färe-Lovell I-O PCU measure ($m^v \geq 1$):*

$$NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) \leq DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) \leq 1. \tag{34}$$

In particular, a sufficient condition for $NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) = DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)$ is that the variable input is a singleton, i.e., $m^v = 1$.

Proposition 3.8 reveals that the biased (sub-vector) radial I-O PCU measure is no smaller than the biased Färe-Lovell I-O PCU measure. This characteristic is valuable because it points out that the result of plant capacity measurement using the (sub-vector) radial input efficiency measure may be biased upward. Proposition 3.8 also suggests the equivalence property between the biased (sub-vector) radial and nonradial I-O PCU measure and its sufficient condition. This property indicates that the biased (sub-vector) radial I-O PCU measure is an unbiased estimation of the biased Färe-Lovell I-O PCU measure when the variable input is a singleton.

In contrast, no such linkage can be established for the unbiased sub-vector radial and nonradial I-O PCU measure. Despite the fact that both numerators ($NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) \leq DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) \leq 1$) and denominators ($NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) \leq DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon) \leq 1$) can be ranked, the ratios of these ($NPCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$ and $PCU_i(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y})$) cannot be ranked accordingly.[8]

Building upon the foundation of these nonradial efficiency measures, it becomes imperative to consider the practical constraints that firms may face when adjusting their input and output levels. This leads us to explore the concept of attainability in Sect. 3.4: this ensures that the proposed capacity measures are not only theoretically sound but also practically feasible. By doing so, we ensure that the capacity measures reflect feasible adjustments in variable input levels, thereby providing a more accurate assessment of capacity utilization.

3.4 Nonradial plant capacity measures: exploration on attainability

We are now in a position to analyse the specification of attainability of the proposed nonradial O-O PCU measure. Johansen (1968) first points out the O-O PCU measure may not be attainable if the additional variable inputs required to achieve maximal output capacity are unavailable. Kerstens et al. (2019b) investigate such attainability issue and define an attainable

[8] $NDF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) \leq DF_i^{SR}(\boldsymbol{x^f}, \boldsymbol{x^v}, \boldsymbol{y}) \leq 1$ can be proven analogous to Proposition 3.8. To save space, its proof is not provided.

O-O PCU measure using a radial output efficiency measure. They also stress that attainability is not a concern for the I-O PCU measure because of free disposability in (variable) inputs. Thus, we only focus on the attainability of a nonradial O-O PCU measure.

Specifically, Kerstens et al. (2019b) define a scalar entitled attainability level for each observation to reflect the restriction on variable inputs. The scalar is consistent with the radial output efficiency measure, but it may be implausible in nonradial plant capacity measures because it imposes tight restrictions on all variable inputs, i.e., the available variable inputs just are proportional to the observed variable inputs for any observation. For instance, both doctors and nurses are quasi-fixed factors (i.e., factors that cannot be expanded rapidly) in hospital capacity measurement. The number of new hiring doctors and nurses often are restricted by other factors like hospital protocol and hospital area, which may not characterize a complete proportional relation to the observed doctors and nurses. Therefore, we explore the attainability issue using more flexible restrictions on variable inputs for the nonradial plant capacity measures.

Recall that the standard axiom of free disposability indicates that the expansion of variable inputs is unlimited. However, in the context of O-O PCU, we clearly observe that this unlimited expansion may lead to a potential attainability issue because of the limited resources in practice. As a potential solution, an attainability level of an observation $\bar{\xi} \in \mathbb{R}_+^{m^v}$ whose component is $\bar{\xi}_i$ is defined as follows.

Definition 3.13 An attainability level $\bar{\xi}$ of observation $(\boldsymbol{x}, \boldsymbol{y})$ is defined as: $\forall \bar{\xi} \in \mathbb{R}_+^{m^v}$, $\exists \xi \in \mathbb{R}_+^{m^v}$ with $\xi \leq \bar{\xi}$ and $\exists \theta \in \mathbb{R}_+^s$ such that $(\boldsymbol{x}^{\boldsymbol{f}}, \xi \odot \boldsymbol{x}^{\boldsymbol{v}}, \theta \odot \boldsymbol{y}) \in T$.

Note that ξ and θ are vectors rather than scalars. The attainability level allows nonproportional reduction or expansion for all variable inputs. In particular, every vector $\bar{\xi} \geq 1$ is feasible to be an attainability level for all observations by the standard axiom of free disposability. An attainability level $\bar{\xi} \leq 1$ might be infeasible for some observations because of variable returns to scale on technology T. However, the attainability level should be selected exogenously to reflect the realistic restriction on all variable inputs. For instance, $\bar{\xi} = (2, 3)$ implies that doubling the first variable input and tripling the second variable input is achievable.

Following the Definition 3.13 of an attainability level, we can now define the attainable Färe-Lovell output efficiency measure as follows.

Definition 3.14 The attainable Färe-Lovell O-O efficiency measure $(ANDF_o)$ at level $\bar{\xi} \in \mathbb{R}_+^{m^v}$ is: $ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \bar{\xi}) = \max\{\frac{1}{s}\sum_{r=1}^{s} \theta_r \mid \theta \odot \boldsymbol{y} \in P(\boldsymbol{x}^{\boldsymbol{f}}, \xi \odot \boldsymbol{x}^{\boldsymbol{v}}), \theta \geq 0, 0 \leq \xi \leq \bar{\xi}\}$.

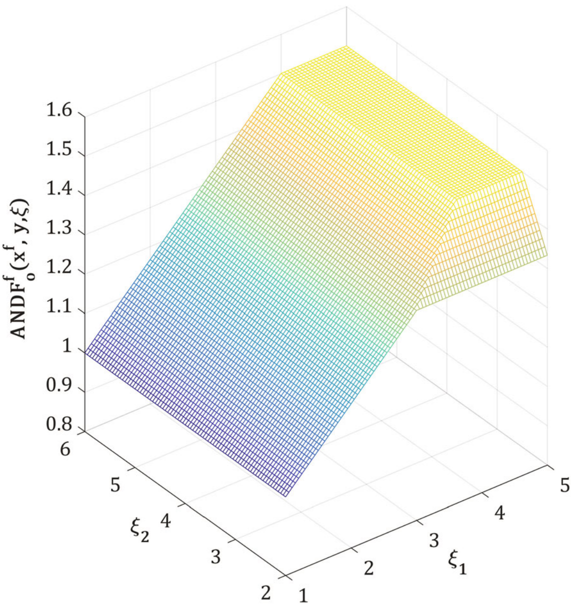
The variable inputs is now restricted by vector $\bar{\xi}$ whose components determine the maximal available variable input in each dimension. Thus, $ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \bar{\xi}) \leq NDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y})$.

We are now in a position to define a new attainable Färe-Lovell O-O PCU measure using the attainable Färe-Lovell O-O efficiency measure in Definition 3.14.

Definition 3.15 An attainable Färe-Lovell O-O PCU $(ANPCU_o)$ at level $\bar{\xi} \in \mathbb{R}_+^{m^v}$ is $ANPCU_o(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \bar{\xi}) = \frac{NDF_o(\boldsymbol{x}, \boldsymbol{y})}{ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \bar{\xi})}$,
where $NDF_o(\boldsymbol{x}, \boldsymbol{y})$ and $ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \bar{\xi})$ are defined previously.

By analogy, $NDF_o(\boldsymbol{x}, \boldsymbol{y})$ is a biased attainable Färe-Lovell PCU measure, but $ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \bar{\xi})$ is an unbiased one. In particular, since $ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \bar{\xi}) \leq$

Fig. 3 Joint effect of ξ_1 and ξ_2 on $ANDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}, \xi)$



$NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$, then $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \bar{\xi}) \geq NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$ always holds. Therefore, the unbiased attainable Färe-Lovell O-O PCU measure is no smaller than the unbiased Färe-Lovell O-O PCU measure. The attainable Färe-Lovell O-O efficiency measure at level $\bar{\xi}$ ($ANDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}, \bar{\xi})$) can be solved by the linear program (B.9) in Appendix B.

We now can report the linkage between the attainable Färe-Lovell O-O PCU measure and the Färe-Lovell O-O PCU measure as follows.

Proposition 3.9 *There exists a set* Δ*, such that for any* $\hat{\hat{\xi}} \in \Delta$*, we have:*

(i) $\forall \xi \geq \hat{\hat{\xi}}$*, we have* $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \xi) = NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$.
(ii) $\forall \xi \leq \hat{\hat{\xi}}$ *and* $\xi \neq \hat{\hat{\xi}}$*, we have* $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \xi) > NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$.

It follows from Proposition 3.9 that there exist multiple thresholds to the convergence of $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \xi)$ to $NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$. In particular, it also implies that $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \xi) = NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$ if the supply of variable inputs is sufficiently large, i.e., $\lim_{\xi \to +\infty} ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \xi) = NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$.

To illustrate the above general result, it can be useful to present a numerical example. To do so, let us consider two observations A and B, where a single fixed input and two variable inputs are employed to generate two outputs. Their production mixes are $(x_A^f, x_{1A}^v, x_{2A}^v, y_{1A}, y_{2A}) = (2, 1, 2, 1, 2)$ and $(x_B^f, x_{1B}^v, x_{2B}^v, y_{1B}, y_{2B}) = (1, 4, 5, 2, 2)$. Suppose $\xi_1 \in [1, 5]$ and $\xi_2 \in [2, 6]$. Take observation A for example, note that $NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$ is not contingent on ξ_1 and ξ_2, we can clearly compute it for observation A as $NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}) = 0.667$ by Definition 3.7. Subsequently,the joint effects of ξ_1 and ξ_2 on $ANDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}, \xi)$ and $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \xi)$ are presented in Figs. 3 and 4, respectively.

Figure 3 illustrates that for any vector $(\xi_1, \xi_2) \geq (4, 2.5)$, $ANDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}, \xi) = 1.5$ always holds; for any vector $(\xi_1, \xi_2) \leq (4, 2.5)$ and $(\xi_1, \xi_2) \neq (4, 2.5)$,

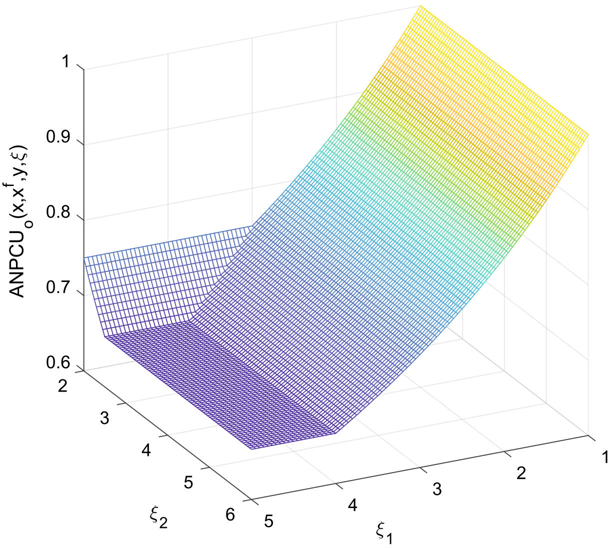


Fig. 4 Joint effect of ξ_1 and ξ_2 on $ANPCU_o(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi)$

$ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi) < 1.5$. It also presents the result that $ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi)$ is non-decreasing as ξ increases in its components ξ_1 and ξ_2, i.e., $\frac{\partial ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi)}{\partial \xi_1} \geq 0$ and $\frac{\partial ANDF_o^f(\boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi)}{\partial \xi_2} \geq 0$.

Figure 4 indicates that for any vector $(\xi_1, \xi_2) \geq (4, 2.5)$, $ANPCU_o^f(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi) = NPCU_o(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}) = 0.667$ always holds; but for any vector $(\xi_1, \xi_2) \leq (4, 2.5)$ and $(\xi_1, \xi_2) \neq (4, 2.5)$, $ANPCU_o^f(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi) > NPCU_o(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}) = 0.667$. It suggest that $\Delta = \{(4, 2.5)\}$ in this numerical example. It is useful to stress that as the dimensionality of variable inputs increases (i.e., $m^v > 2$), then the elements in set Δ increase as well, i.e., Δ is built by more than one elements. Figure 4 also reports the result that $ANPCU_o^f(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi)$ is non-increasing as ξ increases in its components ξ_1 and ξ_2, i.e., $\frac{\partial ANPCU_o^f(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi)}{\partial \xi_1} \leq 0$ and $\frac{\partial ANPCU_o^f(\boldsymbol{x}, \boldsymbol{x}^{\boldsymbol{f}}, \boldsymbol{y}, \xi)}{\partial \xi_2} \leq 0$.

Kerstens and Sadeghi (2024) review the existing PCU concepts and define some new variations. Furthermore, for the radial O-O, I-O, and attainable O-O PCU notions discussed here, they investigate the existence of solutions at both the firm and industry level. While all three PCU concepts exist under variable returns to scale at the firm level, only the I-O PCU measure exists at the industry level in that all firms can simultaneously achieve full capacity. This further emphasizes the importance and practicality of the I-O PCU measure.

4 Empirical illustration

To illustrate how the nonradial capacity measures can be implemented, a secondary data is used to support the replicability of our empirical results. The data selected have been used to illustrate convex and nonconvex I-O technical and economic PCU measures in Kerstens et al.

Table 1 Descriptive statistics of inputs and outputs for French fruit producers

Input/output	Trimmed mean[a]	Minimum	Maximum
Capital (=fixed input)	85,602.58	8891	500,452
Labor (=variable input	229,569	79,569	1,682,201
Materials (=variable input)	157,610.9	19,566	1,523,776
Apple production (output)	2.146273	0.00061	37.98153
Other products (output)	1.37793	0.000672	25.895

[a] 10% trimming level

(2019a).[9] Similarly, we select a three-year panel accounting data of French fruit producers (1984–1986) in a survey from Ivaldi et al. (1996). In particular, we only opt for farms whose production of apples is larger than zero and the orchard's productive land needs to be at least five acres. The three inputs are (i) labor, (ii) capital (including land), and (iii) materials. The two outputs are (i) the apple production, and (ii) an aggregation of remaining products. Detailed definitions of these variables are found in the Appendix 2 of Ivaldi et al. (1996).

Table 1 summarizes the descriptive statistics for the input and output variables of 405 observations: we can infer some heterogeneity in all inputs and outputs. We select capital as the fixed input in the empirical application.

4.1 Comparing radial and nonradial plant capacity notions

Table 2 reports the descriptive statistics for all radial and unweighted nonradial PCU notions.[10] In particular, both convex and nonconvex technologies are taken into account for these PCU notions. The average, the standard deviation, the maximum, and the minimum are presented in Table 2. The results align with the findings presented in the preceding section. For example, in all cases (i.e., radial vs nonradial, convex vs nonconvex), the minima of PCU notions $DF_o(\boldsymbol{x}, \boldsymbol{y})$, $DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$, and $PCU_i(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$ are unity, with the interpretation that unity is the lower bounds of them as shown previously. By analogy, as the upper bounds of PCU notions $PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$, $DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$, and $DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$ are unity, the maxima of them in all cases are unity as shown in Table 2. On average, the radial and nonradial PCU results are rather markedly different. In addition, on average convexity has a non-negligible impact on PCU notions. As a consequence, the outcomes for convex and nonconvex scenarios also exhibit distinct differences.

To further investigate the differences of capacity notions above, we use the Li-test proposed by Li (1996) and modified by Li et al. (2009) in this section. The Li-test is a nonparametric test, which focuses on the differences between entire distributions rather than the first moments (for example, Wilcoxon signed-ranks test). It evaluates whether two kernel-based estimations of density functions $f(x)$ and $g(x)$ (where x is a random variable) differ in a way that is

9 In the current empirical analysis, the attainable O-O PCU measure introduced in Sect. 3.4 is not implemented due to data limitations and the focused scope of the study. Our primary objective is to compare radial and nonradial plant capacity measures under convex and nonconvex technologies. However, recognizing its potential value, we plan to incorporate this procedure in future research utilizing more detailed data sets. This will allow for a comprehensive assessment of plant capacity utilization that fully accounts for operational constraints.

10 While the weighted nonradial plant capacity concepts are more general, in practice it is difficult to come up with a reasonable weight vector.

Table 2 Descriptive statistics for radial and nonradial capacity notions

Convex	$PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$	$DF_o(\boldsymbol{x}, \boldsymbol{y})$	$DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$	$PCU_i(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	$DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$	$DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$
Average	0.7078	3.4883	5.4149	1.7337	0.4233	0.5881
Stand.Dev	0.2204	2.6316	4.6723	1.6340	0.1947	0.1922
Maximum	1.0000	16.2869	35.2953	21.1414	1.0000	1.0000
Minimum	0.0701	1.0000	1.0000	1.0000	0.0473	0.1868
Nonconvex	$PCU_o(x, x^f, y)$	$DF_o(x, y)$	$DF_o^f(x^f, y)$	$PCU_i(x^f, x^v, y)$	$DF_i^{SR}(x^f, x^v, \epsilon)$	$DF_i^{SR}(x^f, x^v, y)$
Average	0.6910	1.6482	2.8910	2.5414	0.4308	0.8309
Stand.Dev	0.2444	1.1113	2.9316	2.1531	0.2019	0.2060
Maximum	1.0000	7.9363	32.4565	21.1414	1.0000	1.0000
Minimum	0.0968	1.0000	1.0000	1.0000	0.0473	0.2696
Convex	$NPCU_o(x, x^f, y)$	$NDF_o(x, y)$	$NDF_o^f(x^f, y)$	$NPCU_i(x^f, x^v, y)$	$NDF_i^{SR}(x^f, x^v, \epsilon)$	$NDF_i^{SR}(x^f, x^v, y)$
Average	0.5678	36.7044	96.6075	2.0886	0.3455	0.5433
Stand.Dev	0.2351	268.8278	793.6409	2.3253	0.1767	0.1899
Maximum	1.0000	4988.9065	13876.9597	30.4766	1.0000	1.0000
Minimum	0.0113	1.0000	1.0000	1.0000	0.0328	0.1751
Nonconvex	$NPCU_o(x, x^f, y)$	$NDF_o(x, y)$	$NDF_o^f(x^f, y)$	$NPCU_i(x^f, x^v, y)$	$NDF_i^{SR}(x^f, x^v, \epsilon)$	$NDF_i^{SR}(x^f, x^v, y)$
Average	0.6025	32.5466	69.1465	3.0599	0.3465	0.7563
Stand.Dev	0.3122	271.4369	592.0259	3.0493	0.1780	0.2341
Maximum	1.0000	4190.7320	10947.3695	30.4766	1.0000	1.0000
Minimum	0.0217	1.0000	1.0000	1.0000	0.0328	0.2290

statistically significant. The null hypothesis of the Li-test is the equality of both density functions almost everywhere, i.e., $H_0 : \forall x, f(x) = g(x)$. The alternative hypothesis states the inequality of both density functions somewhere, i.e., $H_1 : \exists x, f(x) \neq g(x)$. Note that the Li-test can be applied for both dependent and independent variables.[11]

Table 3 reports the Li-test statistics between radial and nonradial capacity notions with convex technology. The Li-test statistic between radial and nonradial capacity notions are reported by components on the diagonal (in bold). For the convex capacity notions, Table 3 indicates that almost all capacity notions follow two by two significantly different contributions at the 1% level, though the Li-test statistics between $NDF_o(\boldsymbol{x}, \boldsymbol{y})$ and $DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$, $NDF_o(\boldsymbol{x}, \boldsymbol{y})$ and $DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$, $NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$ and $DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$ have indistinguishable distributions. The results of components on the diagonal (in bold) imply that radial and nonradial capacity notions follow different contributions with the exception that $DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$ and $NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$ have indistinguishable distributions.

Table 4 presents the Li-test statistics between radial and nonradial capacity notions with nonconvex technology. The Li-test statistic between radial and nonradial capacity notions are reported by components on the diagonal (in bold) again. One observes that all capacity notions follow two by two significantly different distributions at the 1% level. It should be stressed that all radial and nonradial capacity notions follow different contributions on the diagonal (in bold). This results differ from that in the convex case shown in Table 4. Consequently, we can infer that nonconvex technology exhibits a stronger power to discriminate radial and nonradial capacity notions compared with the convex technology. This is linked to the fact mentioned in the introduction that the amounts of slack and surplus variables under convexity is lower than the amounts of slack and surplus variables under nonconvexity.

Wrapping up, radial and nonradial capacity notions present different distributions in both convex and nonconvex cases. This result indicates the overestimation of the optimal capacity of variable inputs, and the underestimation of the optimal capacity of outputs have non-negligible impacts on I-O and O-O capacity notions. Consequently, we suggest that practitioners should reconsider radial capacity notions carefully in practice because of potential biased estimations from radial efficiency measures. Nonradial capacity notions are feasible alternatives because they are generalized plant capacity notions that exhibit more desirable theoretical properties.

4.2 Comparing convex and nonconvex nonradial plant capacity notions

Focusing now on the nonradial plant capacity notions, we compare convex and nonconvex nonradial capacity notions following a similar structure of arguments as in the preceding subsection. By analogy, we implement a Li-test for convex and nonconvex nonradial capacity notions similar to Tables 3 and 4. The Li-test for radial capacity notions is not presented in this subsection as it has been fully investigated in the literature (see for example, Kerstens et al. (2019a)). Table 5 reports the Li-test statistics between convex and nonconvex nonradial capacity notions. The structure of Table 5 is as follows. First, the Li-test statistics between convex and nonconvex nonradial capacity notions are presented by components on the diagonal (in bold). Second, the components above the diagonal is the Li-test statistics between nonconvex capacity notions. Third, the components under the diagonal depict the Li-test statistics between convex capacity notions.

[11] Matlab code for the Li-test based on Li et al. (2009) is available at: `https://github.com/kepiej/DEAUtils`.

Table 3 Li-test statistics between radial and nonradial capacity notions with convex technology

Capacity Notions	$PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$	$DF_o(\boldsymbol{x}, \boldsymbol{y})$	$DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$	$PCU_i(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	$DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$	$DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$
$NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$	**13.969*** **	160.652***	114.528***	143.682***	21.369***	5.735***
$NDF_o(\boldsymbol{x}, \boldsymbol{y})$	136.043***	**9.770*** **	0.380	94.196***	177.154***	−0.746
$NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$	208.641***	37.961 ***	**24.156*** **	168.240***	177.369***	224.635***
$NPCU_i(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	79.201***	31.425***	76.071***	**9.620*** **	194.445***	120.357***
$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$	11.854***	159.408***	180.006***	157.961***	**5.760*** **	47.013***
$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	14.807***	126.901***	165.533***	101.304***	14.557***	**0.149**

Li-test: critical values at 1% level= 2.33($***$); 5% level= 1.64($**$); 10% level= 1.28($*$)

Table 4 Li-test statistics between radial and nonradial capacity notions with nonconvex technology

Capacity Notions	$PCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$	$DF_o(\boldsymbol{x}, \boldsymbol{y})$	$DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$	$PCU_i(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	$DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$	$DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$
$NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$	**8.684*****	33.709***	55.928***	80.142***	26.323***	16.489***
$NDF_o(\boldsymbol{x}, \boldsymbol{y})$	53.903***	**65.087*****	5.727***	24.660***	169.102***	84.934***
$NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$	161.607***	144.837***	**40.709*****	56.264***	192.351***	176.239***
$NPCU_i(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	106.158***	97.492***	37.305***	**24.508*****	80.495***	112.766***
$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$	74.273***	115.568***	150.205***	141.004***	**6.164*****	92.419***
$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	3.062***	17.532***	36.462***	22.808***	62.284***	**3.626*****

Li-test: critical values at 1% level= 2.33($***$); 5% level= 1.64($**$); 10% level= 1.28($*$)

Table 5 Li-test statistics between convex and nonconvex nonradial capacity notions

Capacity Notions	$NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$	$NDF_o(\boldsymbol{x}, \boldsymbol{y})$	$NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$	$NPCU_i(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$	$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$
$NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$	**15.922*** **	84.790***	154.391***	107.789***	26.944***	11.551***
$NDF_o(\boldsymbol{x}, \boldsymbol{y})$	104.884***	**6.820*** **	19.329***	25.570***	155.881***	61.053***
$NDF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$	206.798***	19.663***	**7.823*** **	47.887***	206.196***	162.130***
$NPCU_i(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	207.306***	72.056***	139.418***	**19.151*** **	139.833***	11.234***
$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$	41.402***	189.304***	238.425***	176.377***	**−1.887** **	85.650***
$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$	8.466***	−0.132	228.188***	159.735***	43.890***	**26.270*** **

Li-test: critical values at 1% level= 2.33($* * *$); 5% level= 1.64($**$); 10% level= 1.28($*$)

The following conclusions can be drawn from Table 5. First, the components on the diagonal (in bold) suggest that capacity notions under convex and nonconvex technologies follow different distributions. One exception is that the Li-test statistics of $NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ is marginally significant at the 5% level. Second, the components above the diagonal show that all nonconvex capacity notions exhibit significantly different distributions when compared two by two. Third, the components under the diagonal reporting all convex capacity notions exhibit significantly different distributions when compared two by two, apart from the Li-test statistics between $NDF_o(\boldsymbol{x}, \boldsymbol{y})$ and $NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y})$ where the two convex capacity notions have indistinguishable distributions.

5 Conclusions

This main focus of this contribution is on defining PCU notions based on nonradial rather than traditional radial efficiency measures. These alternative nonradial PCU concepts are based on nonradial weighted Färe-Lovell efficiency measures. First, we define the traditional radial O-O PCU, the new partial O-O PCU, and the rather recent I-O PCU notions. Then, we use graphical illustrations to shed light on the possibility that radial PCU notions may well leave ample amounts of slacks or unmeasured inefficiency. This underscores the need for more comprehensive nonradial measures that can accurately capture inefficiencies in multi-dimensional production environments. Thereafter, we have first formally defined weighted and unweighted Färe-Lovell O-O PCU notions and we explore their main properties. We also formally define weighted and unweighted Färe-Lovell I-O PCU notions and we consider their properties.

Finally, we also investigate how the introduction of vectors of nonradial attainability levels rather than a simple scalar specification can render the attainable O-O PCU concept proposed by Kerstens et al. (2019b) more flexible. This enhancement allows for a more nuanced assessment of capacity utilization under varying operational constraints.

The empirical section makes use of a secondary data set of French fruit producers and finds pertinent differences between radial and nonradial plant capacity notions. Additionally, it demonstrates the significance of nonradial plant capacity ideas especially in the context of a nonconvex technology. These empirical findings validate the theoretical advantages of our nonradial models, showcasing their effectiveness in capturing capacity utilization more accurately than traditional radial measures. These results provide convincing arguments to the non-negligible role of nonconvexity in nonradial plant capacity notions. Thus, practitioners should reconsider imposing the convexity axiom in plant capacity measurement because convexity is clearly not harmless.

One avenue for future research is to expand on the application of nonradial efficiency measures towards the long-run O-O and I-O PCU measures proposed in Cesaroni et al. (2019). Current empirical applications of these PCU notions (see, for instance, Kerstens and Shen (2021), Shen et al. (2022), or Song et al. (2023)) rely so far on radial efficiency measures only. Also an empirical application of the nonradial attainable O-O PCU measure introduced in Sect. 3.4 remains to be developed.

Appendices: Supplementary material

A Proofs

Proposition A.1 *The maximal output capacity $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ has the following properties:*

(i) It belongs to the isoquant of $P^f(\boldsymbol{x}^f)$, i.e., $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ Isoq $P^f(\boldsymbol{x}^f)$.
(ii) It belongs to the isoquant of $P(\boldsymbol{x}^f, +\infty)$, i.e., $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ Isoq $P(\boldsymbol{x}^f, +\infty)$.

Proof First, suppose $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \notin$ Isoq $P^f(\boldsymbol{x}^f)$, then there exists $\theta \in (1, \infty)$, such that $\theta \boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in P^f(\boldsymbol{x}^f)$. As $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} = DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})\boldsymbol{y}$ (see Definition 3.5), we obtain $\theta DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})\boldsymbol{y} \in P^f(\boldsymbol{x}^f)$. Let $\theta^* = \theta DF_o^f(\boldsymbol{x}^f, \boldsymbol{y}) > DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$, $\theta^* y \in P^f(\boldsymbol{x}^f)$ holds. As a consequence, there exists a feasible solution $\theta^* (> DF_o^f(\boldsymbol{x}^f, \boldsymbol{y}))$ to Program (8). Therefore, $DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$ is not the optimal solution of Program (8), which contradicts to the the definition of $DF_o^f(\boldsymbol{x}^f, \boldsymbol{y})$ as shown in (8). Hence, $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ Isoq $P^f(\boldsymbol{x}^f)$.

Second, to prove $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ Isoq $P(\boldsymbol{x}^f, +\infty)$, we only need to prove Isoq $P^f(\boldsymbol{x}^f) =$ Isoq $P(\boldsymbol{x}^f, +\infty)$ as $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ Isoq $P^f(\boldsymbol{x}^f)$. Recall that $P(\boldsymbol{x}^f, +\infty) = \{\boldsymbol{y} \mid (\boldsymbol{x}^f, +\infty, \boldsymbol{y}) \in T\}$ and $T^f = \{(\boldsymbol{x}^f, \boldsymbol{y}) \mid (\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y}) \in T\}$, $P(\boldsymbol{x}^f, +\infty)$ can be reformulated as $P(\boldsymbol{x}^f, +\infty) = \{\boldsymbol{y} \mid (\boldsymbol{x}^f, \boldsymbol{y}) \in T^f\}$ because of free disposability in variable inputs (i.e., $\boldsymbol{x}^v < +\infty$). Combining $P^f(\boldsymbol{x}^f) = \{\boldsymbol{y} \mid (\boldsymbol{x}^f, \boldsymbol{y}) \in T^f\}$, we have $P(\boldsymbol{x}^f, +\infty) = P^f(\boldsymbol{x}^f)$. Consequently, Isoq $P(\boldsymbol{x}^f, +\infty) =$ Isoq $P^f(\boldsymbol{x}^f)$. Thus, $\boldsymbol{y}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ Isoq $P(\boldsymbol{x}^f, +\infty)$. □

Proposition A.2 *The minimal input capacity $\boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}$ with the fixed inputs $\boldsymbol{x}^f$ belongs to the isoquant of $L(\epsilon)$, i.e., $(\boldsymbol{x}^f, \boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}) \in$ Isoq $L(\epsilon)$.*

Proof Suppose $(\boldsymbol{x}^f, \boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}) \notin$ Isoq $L(\epsilon)$, then there exists $\beta \in [0, 1)$ such that $\beta(\boldsymbol{x}^f, \boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}) \in L(\epsilon)$. By the assumption of free disposability in (variable) inputs, we have $(\boldsymbol{x}^f, \beta\boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)} \in L(\epsilon)$ because of $\boldsymbol{x}^f > \beta\boldsymbol{x}^f$. As $\boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)} = DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)\boldsymbol{x}^v$ (see Definition 3.6), we obtain $(\boldsymbol{x}^f, \beta DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)\boldsymbol{x}^v \in L(\epsilon))$. Let $\beta^* = \beta DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon) < DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$,$(\boldsymbol{x}^f, \beta^*\boldsymbol{x}^v) \in L(\epsilon)$ holds. Consequently, there exists a feasible solution $\beta^* < DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ to Program (7). Hence, $DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ is not the optimal solution of Program (7) when $\boldsymbol{y} = \epsilon$, which contradicts to the definition of $DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ as shown in (7). Therefore, $(\boldsymbol{x}^f, \boldsymbol{x^v}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}) \in$ Isoq $L(\epsilon)$. □

Proposition A.3 *The optimal output capacity $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$ has the following properties:*

(i) It pertains to the efficient subset of $P^f(\boldsymbol{x}^f)$, i.e., $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ Eff $P^f(\boldsymbol{x}^f)$.
(ii) It pertains to the efficient subset of $P(\boldsymbol{x}^f, +\infty)$, i.e., $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \in$ Eff $P(\boldsymbol{x}^f, +\infty)$.

Proof First, suppose $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} \notin$ Eff $P^f(\boldsymbol{x}^f)$, then there exists $\boldsymbol{y}' \geq \boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$, $\boldsymbol{y}' \neq \boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$, such that $\boldsymbol{y}' \in P^f(\boldsymbol{x}^f)$. Since $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})} = \theta^* \odot \boldsymbol{y}$ (see Definition 3.9), we obtain $\theta^* \odot \boldsymbol{y} + \boldsymbol{z} \in P^f(\boldsymbol{x}^f)$ where $\boldsymbol{z} \in \mathbb{R}^r_+ = \boldsymbol{y}' - \boldsymbol{y}^{WN}_{o,(\boldsymbol{x}^f,\boldsymbol{y})}$. Suppose the tth element of vector $\boldsymbol{z}$ denoted by z_t, $t = 1, ..., s$ is strictly greater than zero while other elements are equal to zero, then we get $(\theta_1^* y_1, ..., \theta_t^* y_t + z_t, ..., \theta_s^* y_s) \in P^f(\boldsymbol{x^f})$. Let $\theta_t' = \frac{z_t}{y_t} > 0$ and $\theta_t^{**} = \theta_t' + \theta_t^* > \theta_t^*$, we have $(\theta_1^* y_1, ..., \theta_t^{**} y_t, ..., \theta_s^* y_s) \in P^f(\boldsymbol{x^f})$. Thus, $(\theta_1^*, ..., \theta_t^{**}, ..., \theta_s^*)$ is a feasible solution to Program (19), the corresponding value of objective function is $\sum_{r=1, r\neq t}^{s} \mu_r \theta_r^* + \mu_t \theta_t^{**}$. Since $\sum_{r=1, r\neq t}^{s} \mu_r \theta_r^* + \mu_t \theta_t^{**} > \sum_{r=1, r\neq t}^{s} \mu_r \theta_r^* + \mu_t \theta_t^{**} = \sum_{r=1}^{s} \mu_r \theta_r^*$, $(\theta_1^*, ..., \theta_t^*, ..., \theta_s^*)$ is not the optimal solution of Program (19), which contracts to the definition of $WNDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$ as shown in (19). Hence, $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x^f}, \boldsymbol{y})} \in \text{Eff } P^f(\boldsymbol{x^f})$.

Second, as $P(\boldsymbol{x^f}, +\infty) = P^f(\boldsymbol{x^f})$ (see the proof of Proposition 3.1), it is obvious that Eff $P(\boldsymbol{x^f}, +\infty) = \text{Eff } P^f(\boldsymbol{x^f})$. As a consequence, $\boldsymbol{y}^{WN}_{o,(\boldsymbol{x^f}, \boldsymbol{y})} \in \text{Eff } P(\boldsymbol{x^f}, +\infty)$. □

Proposition A.4 *The generalized framework for the biased O-O PCU measure is defined as:*

$$GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = \max\{\sum_{r=1}^{s} \mu_r \theta_r \mid \theta \odot \boldsymbol{y} \in P^f(\boldsymbol{x^f}), \mu \in \Lambda, \theta \in \Gamma\}. \quad \text{(A.1)}$$

whereby:

(i) $\Lambda = \Lambda^1 = \{\mu \mid \mu_1 = \mu_2 = \cdots = \mu_s = \frac{1}{s}\}$, $\Gamma = \Gamma^1 = \{\theta \mid \theta_1 = \theta_2 = \cdots = \theta_s \geq 1\} \Rightarrow GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = DF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$;
(ii) $\Lambda = \Lambda^2 = \{\mu \mid \mu_r = 1, \mu_{-r} = 0\}$, $\Gamma = \Gamma^2 = \{\theta \mid \theta_r \geq 1, \theta_{-r} = 1\} \Rightarrow GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})$;
(iii) $\Lambda = \Lambda^3 = \{\mu \mid \sum_{r=1}^{s} \mu_r = 1, \mu_r > 0, r = 1, ..., s\}$, $\Gamma = \Gamma^3 = \{\theta \mid \theta \geq 1\} \Rightarrow GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = WNDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$;
(iv) $\Lambda = \Lambda^1 = \{\mu \mid \mu_1 = \mu_2 = \cdots = \mu_s = \frac{1}{s}\}$, $\Gamma = \Gamma^3 = \{\theta \mid \theta \geq 1\} \Rightarrow GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$;

Proof First, when $\mu_1 = \mu_2 = \cdots = \mu_s = \frac{1}{s}$, let $\bar{\theta} = \theta_1 = \theta_2 = \cdots = \theta_s \geq 1$, we have $GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = \max\{\bar{\theta} \mid \bar{\theta}\boldsymbol{y} \in P^f(\boldsymbol{x^f}), \bar{\theta} \in [1, +\infty)\} = DF_o^f(\boldsymbol{x^f}, y)$.

Second, when $\mu_r = 1, \mu_{-r} = 0$, and $\theta_r \geq 1, \theta_{-r} = 1$, we have $GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = \max\{\theta_r \mid (\theta_r y_r, \boldsymbol{y_{-r}}) \in P^f(\boldsymbol{x^f}), \theta_r \in [1, +\infty)\} = DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})$.

Third, when $\sum_{r=1}^{s} \mu_r = 1, \mu_r > 0, r = 1, ..., s$ and $\theta \geq 1$, $GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = WNDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$ by (19).

Fourth, when $\mu_1 = \mu_2 = \cdots = \mu_s = \frac{1}{s}$, $GDF_o^f(\boldsymbol{x^f}, \boldsymbol{y} \mid \Lambda, \Gamma) = NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$ by (16). □

Proposition A.5 *The following linkages can be established among biased radial O-O PCU measure, partial O-O PCU measure, and Färe-Lovell O-O PCU measure ($s \geq 1$):*

$$1 \leq DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) \leq DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) \leq NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}), r = 1, ..., s. \quad \text{(A.2)}$$

In particular,

(i) a sufficient condition for $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) < NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}), r = 1, ..., s$ *is that* $\boldsymbol{y} \notin$ *Eff* $P^f(\boldsymbol{x^f})$, *i.e.,* $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) > 1$;

(ii) a sufficient condition for $DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})$ *is that output is a singleton, i.e.,* $s = 1$.

Proof First, $DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) \geq 1$ is satisfied by definition (see measure (8)).

Second, let $\theta^* \geq 1$ be the optimal solution to program $\max\{\theta \mid \theta \boldsymbol{y} \in P^f(\boldsymbol{x^f}), \theta \in [1, +\infty)\}$, i.e., $\theta^* = DF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$, then we have $(\theta^* y_r, \theta^* \boldsymbol{y_{-r}}) \in P^f(\boldsymbol{x^f})$. By the assumption of free disposability in outputs, we obtain $(\theta^* y_r, \boldsymbol{y_{-r}}) \in P^f(\boldsymbol{x^f})$ due to $\boldsymbol{y_{-r}} \leq \theta^* \boldsymbol{y_{-r}}$. Thus, θ^* is a feasible solution to program $\max\{\theta_r \mid (\theta_r y_r, \boldsymbol{y_{-r}}) \in P^f(\boldsymbol{x^f}), \theta_r \in [1, +\infty)\}, r = 1, ..., s$, thereby $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) \geq \theta^*$ (see Definition 3.2), i.e., $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) \geq DF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$.

Third, to prove $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) \leq NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}), r = 1, ..., s$, we should consider the following two cases: (i) $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = 1$ and (ii) $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) > 1$. In the former case, we have $\theta_1^* = ... = \theta_s^* = 1$ where θ^* is the optimal solution of Program (16) whose component is $\theta_r^*, r = 1, ..., s$. Clearly, $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) = 1$ must be satisfied according Definition 3.2, which can be proven by contradiction as follows. Suppose $\theta_r^* = 1$ is not the optimal solution of Program (11), then there exists $\theta_r^{**} > 1$ such that $(\theta_r^{**} y_r, \boldsymbol{y_{-r}}) \in P^f(\boldsymbol{x^f})$. Thus, $(1_1, ..., \theta_r^{**}, ..., 1_s)$ is a feasible solution to Program (16). The corresponding value of objective function is $\frac{s-1+\theta_r^{**}}{s} > 1$ because of $\theta_r^{**} > 1$, thereby $\theta_1^* = ... = \theta_s^* = 1$ is not the optimal of Program (16) and $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) \neq 1$, which contradicts to $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = 1$. Hence, we obtain $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}), r = 1, ..., s$. In the latter case, we consider the following two subcases (ii-1) $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) = 1$ and (ii-2) $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) > 1$. In sub-case (ii-1), $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) > DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})$ holds clearly. In sub-case (ii-2), suppose θ_r' is the optimal solution of Program (11), i.e., $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) = \theta_r'$. Then $(1_1, ..., \theta_r', ..., 1_s)$ is a feasible solution to Program (16), from which we obtain $\frac{s-1+\theta_r'}{s} \leq NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$. Reformulate the formula above, we get $\frac{NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})-1}{\theta_r'-1} \geq \frac{1}{s}$. Therefore, $\frac{NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})-1}{DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})-1} > 0$, i.e., $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) > DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})$. Wrapping up, $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) \leq NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}), r = 1, ..., s$.

Fourth, when $NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) > 1$, $DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}}) < NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}), r = 1, ..., s$ have been proven above (see subcases (ii-1) and (ii-2)).

Finally, when the output is a singleton, $DF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}) = DF_{o(r)}^f(\boldsymbol{x^f}, y_r, \boldsymbol{y_{-r}})$ always holds by definition. □

Proposition A.6 *The optimal input capacity* $\boldsymbol{x}_{i,(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)}^{v,WN}$ *with the fixed inputs* $\boldsymbol{x^f}$ *belongs to the isoquant of* $L(\epsilon)$, *i.e.,* $(\boldsymbol{x^f}, \boldsymbol{x}_{i,(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)}^{v,WN}) \in$ *Isoq* $L(\epsilon)$.

Proof Suppose $(\boldsymbol{x^f}, \boldsymbol{x}_{i,(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)}^{v,WN}) \notin$ Isoq $L(\epsilon)$, then there exists $\beta \in [0, 1)$ such that $\beta(\boldsymbol{x^f}, \boldsymbol{x}_{i,(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)}^{v,WN}) \in L(\epsilon)$. By the assumption of free disposability in (variable) inputs, we can induce that $(\boldsymbol{x^f}, \beta \boldsymbol{x}_{i,(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)}^{v,WN} \in L(\epsilon)$ due to $\boldsymbol{x^f} > \beta \boldsymbol{x^f}$. Since $\boldsymbol{x}_{i,(\boldsymbol{x^f}, \boldsymbol{x^v}, \epsilon)}^{v,WN} = \beta^* \odot \boldsymbol{x^v}$ (see Definition 3.12), we obtain $(\boldsymbol{x^f}, \beta \cdot \beta^* \odot \boldsymbol{x^v}) \in L(\epsilon)$. Let $\beta^{**} = \beta \cdot \beta^* < \beta^*$, then $(\boldsymbol{x^f}, \beta^{**} \odot \boldsymbol{x^v}) \in L(\epsilon)$. As a consequence, there exists a feasible solution $\beta^{**} < \beta^*$ to Program (29) whose corresponding value of objective function is $\sum_{i=1}^{m^v} \eta_i \beta^{**}$. β^* is not the optimal solution of Program (29) because of $\sum_{i=1}^{m^v} \eta_i \beta^{**} < \sum_{i=1}^{m^v} \eta_i \beta^*$, contradicting to the definition of $WNDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ as shown in (29). Therefore, $(\boldsymbol{x}^f, \boldsymbol{x}^{v,WN}_{i,(\boldsymbol{x}^f,\boldsymbol{x}^v,\epsilon)}) \in \text{Isoq } L(\epsilon)$. □

Proposition A.7 *The generalized framework for the biased I-O PCU measure is defined as:*

$$GDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon \mid \Upsilon, \Phi) = \min\{\sum_{i=1}^{m^v} \eta_i \beta_i \mid (\boldsymbol{x}^f, \beta \odot \boldsymbol{x}^v) \in L(\epsilon), \eta \in \Upsilon, \beta \in \Phi\} \quad (A.3)$$

where

(i) $\Upsilon = \Upsilon^1 = \{\eta \mid \eta_1 = \eta_2 = \cdots = \eta_{m^v} = \frac{1}{m^v}\}$, $\Phi = \Phi^1 = \{\beta \mid 0 \le \beta_1 = \beta_2 = \cdots = \beta_{m^v} \le 1\} \Rightarrow GDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon \mid \Upsilon, \Phi) = DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$;

(ii) $\Upsilon = \Upsilon^2 = \{\eta \mid \sum_{i=1}^{m^v} \eta_i = 1, \eta_i > 0, i = 1, ..., m^v\}$, $\Phi = \Phi^2 = \{\beta \mid 0 \le \beta \le 1\} \Rightarrow GDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon \mid \Upsilon, \Phi) = WNDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$.

(iii) $\Upsilon = \Upsilon^1 = \{\eta \mid \eta_1 = \eta_2 = \cdots = \eta_{m^v} = \frac{1}{m^v}\}$, $\Phi = \Phi^2 = \{\beta \mid 0 \le \beta \le 1\} \Rightarrow GDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon \mid \Upsilon, \Phi) = NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$.

Proof First, when $\eta_1 = \eta_2 = \cdots = \eta_{m^v} = \frac{1}{m^v}$, let $\bar{\beta} = \beta_1 = \beta_2 = \cdots = \beta_{m^v}$, we have $GDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon \mid \Upsilon, \Phi) = \min\{\bar{\beta} \mid (\boldsymbol{x}^f, \bar{\beta}\boldsymbol{x}^v) \in L(\epsilon), \bar{\beta} \in [0, 1]\} = DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$.

Second, when $\sum_{i=1}^{m^v} \eta_i = 1, \eta_i > 0, i = 1, ..., m^v$ and $0 \le \beta \le 1$, $GDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon \mid \Upsilon, \Phi) = WNDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ holds by (29).

Third, when $\eta_1 = \eta_2 = \cdots = \eta_{m^v} = \frac{1}{m^v}$, $GDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon \mid \Upsilon, \Phi) = NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ holds by (25).

Proposition A.8 *The following linkage can be established between the biased (sub-vector) radial I-O PCU measure and the biased Färe-Lovell I-O PCU measure ($m^v \ge 1$):*

$$NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon) \le DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon) \le 1. \quad (A.4)$$

In particular, a sufficient condition for $NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon) = DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ is that the variable input is a singleton, i.e., $m^v = 1$.

Proof First, $DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon) \le 1$ is satisfied by setting $\boldsymbol{y} = \epsilon$ in measure (7).

Second, let $\beta^* \in [0, 1]$ be the optimal solution of program $\min\{\beta \mid (\boldsymbol{x}^f, \beta\boldsymbol{x}^v) \in L(\epsilon), \beta \in [0, 1]\}$, i.e., $\beta^* = DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$, then we get $\beta_1 = \beta_2 = \cdots = \beta_{m^v} = \beta^*$ is a feasible solution to program $\min\{\frac{1}{m^v} \sum_{i=1}^{m^v} \beta_i \mid (\boldsymbol{x}^f, \beta \odot \boldsymbol{x}^v) \in L(\epsilon), \beta_i \in [0, 1]\}$. The corresponding value of objective function is β^*. Hence, $\beta^* \ge NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$ (see measure (25)), i.e., $NDF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon) \le DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \epsilon)$.

Proposition A.9 *There exists a set Δ, such that for any $\hat{\xi} \in \Delta$, we have:*

(i) $\forall \xi \ge \hat{\xi}$, *we have* $ANPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}, \xi) = NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$.

(ii) $\forall \xi \le \hat{\xi}$ *and* $\xi \ne \hat{\xi}$, *we have* $ANPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y}, \xi) > NPCU_o(\boldsymbol{x}, \boldsymbol{x}^f, \boldsymbol{y})$.

Proof $ANDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}, \hat{\xi})$ is non-decreasing with $\hat{\xi}$ because the feasible region of Model (B.9) is enlarged as $\hat{\xi}$ increases. Combined with $ANDF_o^f(\boldsymbol{x^f}, \boldsymbol{y}, \hat{\xi}) \leq NDF_o^f(\boldsymbol{x^f}, \boldsymbol{y})$, we clearly obtain that there exists a set Δ, such that for any $\hat{\xi} \in \Delta$, we have (i) $\forall \xi \geq \hat{\xi}$, $ANDF_o(\boldsymbol{x^f}, \boldsymbol{y}, \xi) = NDF_o(\boldsymbol{x^f}, \boldsymbol{y})$; and (ii) $\forall \xi \leq \hat{\xi}$ and $\xi \neq \hat{\xi}$, $ANDF_o(\boldsymbol{x^f}, \boldsymbol{y}, \xi) < NDF_o(\boldsymbol{x^f}, \boldsymbol{y})$. By the definitions of $NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$ and $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \bar{\xi})$ (see Definitions 3.7 and 3.15), we can induce that for any $\hat{\xi} \in \Delta$, we have (i)$\forall \xi \geq \hat{\xi}$, $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \xi) = NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$; and (ii) $\forall \xi \leq \hat{\xi}$ and $\xi \neq \hat{\xi}$, $ANPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y}, \xi) > NPCU_o(\boldsymbol{x}, \boldsymbol{x^f}, \boldsymbol{y})$.

B Computing nonradial plant capacity notions

This appendix presents the estimation of various capacity concepts' components within a non-parametric frontier framework, assuming VRS. To formulate the models, we will first review the notations introduced in this contribution. The vector of m inputs, denoted as $\boldsymbol{x} \in \mathbb{R}_+^m$, is capable of generating a vector of s outputs, denoted as $\boldsymbol{y} \in \mathbb{R}_+^s$. The input vector $\boldsymbol{x}$ can be divided into two parts: a fixed component ($\boldsymbol{x^f}$) and a variable component ($\boldsymbol{x^v}$), represented as $\boldsymbol{x} = (\boldsymbol{x^f}, \boldsymbol{x^v})$. For each observed production unit k under evaluation, and the corresponding output vector ($\boldsymbol{y_k}$) are known. The fixed and variable input components for the unit are denoted as $\boldsymbol{x_k^f}$ and $\boldsymbol{x_k^v}$, respectively. Lastly, since non-parametric frontier technologies are based on activity analysis, we require a vector of activity variables, $\lambda = (\lambda_1, \ldots, \lambda_n)$, which indicates the intensity levels at which each of the n observed activities is conducted.

Note that to save space, we only present linear programs of efficiency and plant capacity measures under C. The efficiency and plant capacity measures under NC can be computed by adding the binary integer constraint $\lambda_j \in \{0, 1\}$ to the linear programs under C.

Using nonparametric frontier technologies, one can obtain the weighted Färe-Lovell output efficiency measure relative to production correspondence $P(\boldsymbol{x})$ for an evaluated observation $(\boldsymbol{x_k}, \boldsymbol{y_k}), k = 1, ..., n$ as

$$
\begin{aligned}
WNDF_o(\boldsymbol{x_k}, \boldsymbol{y_k}) = \max_{\theta_r, \lambda_j} \; & \sum_{r=1}^{s} \mu_r \theta_r \\
s.t \; & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ik}, i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \theta_r y_{rk}, r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \theta_r \geq 1, \lambda_j \geq 0, r = 1, ..., s, j = 1, ..., n.
\end{aligned}
\tag{B.1}
$$

By analogy, the weighted Färe-Lovell output efficiency measure relative to production correspondence $P^f(\boldsymbol{x^f})$ observation $(\boldsymbol{x_k}, \boldsymbol{y_k}), k = 1, ..., n$ is computed as

$$WNDF_o^f(\boldsymbol{x_k^f}, \boldsymbol{y_k}) = \max_{\theta_r, \lambda_j, x_k^v} \sum_{r=1}^{s} \mu_r \theta_r$$
$$\begin{aligned} s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^f \le x_{ik}^f, i = 1, ..., m^f, \\ & \sum_{j=1}^{n} \lambda_j x_{ij}^v \le x_k^v, i = 1, ..., m^v, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge \theta_r y_{rk}, r = 1, ..., s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & x_k^v \ge 0, \theta_r \ge 1, \lambda_j \ge 0, r = 1, ..., s, j = 1, ..., n. \end{aligned} \tag{B.2}$$

The Färe-Lovell output efficiency measures relative to production correspondence $P(\boldsymbol{x})$ for observation $(\boldsymbol{x_k}, \boldsymbol{y_k}), k = 1, ..., n$ as

$$NDF_o(\boldsymbol{x_k}, \boldsymbol{y_k}) = \max_{\theta_r, \lambda_j} \frac{1}{s} \sum_{r=1}^{s} \theta_r$$
$$\begin{aligned} s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ik}, i = 1, ..., m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge \theta_r y_{rk}, r = 1, ..., s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \theta_r \ge 1, \lambda_j \ge 0, r = 1, ..., s, j = 1, ..., n. \end{aligned} \tag{B.3}$$

By analogy, the Färe-Lovell output efficiency measure relative to production correspondence $P^f(\boldsymbol{x^f})$ observation $(\boldsymbol{x_k}, \boldsymbol{y_k}), k = 1, ..., n$ is computed as

$$NDF_o^f(\boldsymbol{x_k^f}, \boldsymbol{y_k}) = \max_{\theta_r, \lambda_j, x_k^v} \frac{1}{s} \sum_{r=1}^{s} \theta_r$$
$$\begin{aligned} s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^f \le x_{ik}^f, i = 1, ..., m^f, \\ & \sum_{j=1}^{n} \lambda_j x_{ij}^v \le x_k^v, i = 1, ..., m^v, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge \theta_r y_{rk}, r = 1, ..., s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & x_k^v \ge 0, \theta_r \ge 1, \lambda_j \ge 0, r = 1, ..., s, j = 1, ..., n. \end{aligned} \tag{B.4}$$

We can obtain the sub-vector weighted Färe-Lovell input efficiency measure relative to input correspondence $L(\boldsymbol{y})$ for observation $(\boldsymbol{x_k}, \boldsymbol{y_k})$ as:

$$
\begin{aligned}
WNDF_i^{SR}(\boldsymbol{x_k^f}, \boldsymbol{x_k^v}, \boldsymbol{y_k}) = \min_{\beta_i, \lambda_j} & \sum_{i=1}^{m^v} \eta_i \beta_i \\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^f \leq x_{ik}^f, i = 1, ..., m^f, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^v \leq \beta_i x_{ik}^v, i = 1, ..., m^v, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}, r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& 0 \leq \beta_i \leq 1, \lambda_j \geq 0, i = 1, ..., m^v, j = 1, ..., n.
\end{aligned}
\tag{B.5}
$$

By analogy, the sub-vector weighted Färe-Lovell input efficiency measure relative to input correspondence $L(\epsilon)$ is computed as:

$$
\begin{aligned}
WNDF_i^{SR}(\boldsymbol{x_k^f}, \boldsymbol{x_k^v}, \epsilon) = \min_{\beta_i, \lambda_j} & \sum_{i=1}^{m^v} \eta_i \beta_i \\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^f \leq x_{ik}^f, i = 1, ..., m^f, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^v \leq \beta_i x_{ik}^v, i = 1, ..., m^v, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \epsilon, r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& 0 \leq \beta_i \leq 1, \lambda_j \geq 0, i = 1, ..., m^v, j = 1, ..., n.
\end{aligned}
\tag{B.6}
$$

One can obtain the sub-vector Färe-Lovell input efficiency measure relative to input set $L(\boldsymbol{y})$ for observation $(\boldsymbol{x_k}, \boldsymbol{y_k})$ as:

$$
\begin{aligned}
NDF_i^{SR}(\boldsymbol{x_k^f}, \boldsymbol{x_k^v}, \boldsymbol{y_k}) = \min_{\beta_i, \lambda_j} & \frac{1}{m^v} \sum_{i=1}^{m^v} \beta_i \\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^f \leq x_{ik}^f, i = 1, ..., m^f, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^v \leq \beta_i x_{ik}^v, i = 1, ..., m^v, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}, r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& 0 \leq \beta_i \leq 1, \lambda_j \geq 0, i = 1, ..., m^v, j = 1, ..., n.
\end{aligned}
\tag{B.7}
$$

By analogy, the biased Färe-Lovell I-O PCU measure for observation $(\boldsymbol{x_k}, \boldsymbol{y_k})$ is computed as:

$$
\begin{aligned}
NDF_i^{SR}(\boldsymbol{x_k^f}, \boldsymbol{x_k^v}, \epsilon) = \min_{\beta_i, \lambda_j} \; & \frac{1}{m^v} \sum_{i=1}^{m^v} \beta_i \\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^f \leq x_{ik}^f, i = 1, ..., m^f, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^v \leq \beta_i x_{ik}^v, i = 1, ..., m^v, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \epsilon, r = 1, ..., s, \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& 0 \leq \beta_i \leq 1, \lambda_j \geq 0, i = 1, ..., m^v,
\end{aligned} \tag{B.8}
$$

We can model the attainable Färe-Lovell output efficiency measure at level $\bar{\xi}$ as

$$
\begin{aligned}
ANDF_o^f(\boldsymbol{x_k^f}, \boldsymbol{y_k}, \bar{\xi}) = \max_{\theta_r, \lambda_j} \; & \frac{1}{s} \sum_{r=1}^{s} \theta_r \\
s.t \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^f \leq x_{ik}^f, i = 1, ..., m^f, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^v \leq x_i^v, i = 1, ..., m^v, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \theta_r y_{rk}, r = 1, ..., s, \\
& x_i^v \leq \bar{\xi}_i x_{ik}^v, i = 1, ..., m^v, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \theta_r \geq 0, \lambda_j \geq 0, r = 1, ..., s, j = 1, ..., n.
\end{aligned} \tag{B.9}
$$

The constraint $x_i^v \leq \bar{\xi}_i x_{ik}^v, i = 1, ..., m^v$ establishes a link between the observed ith variable input and the decision variable x_i^v via $\bar{\xi}_i$.

Data availability The data used in this study are available in the Journal of Applied Econometrics Data Archive: `http://qed.econ.queensu.ca/jae/1996-v11.6/ivaldi-ladoux-ossard-simioni/`.

Declarations

Conflict of interest Statement The authors report there are no Conflict of interest to declare.

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